Birzeit University

Mathematics Department

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Course Code: MATH234

Title: Linear Algebra

CH1: Matrices and Systems of Equations

1.1 System of linear Equations.

Df. A linear equation in n unknowns is an equation of the form

$$\boxed{a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = b,}$$

where $a_1, a_2, \cdots, a_n$ and b are real numbers, and $x_1, x_2, \cdots, x_n$ are variables.

Ex. $ax+by=c$ is a linear equation of two variables (x and y), a, b, and c are constants.

Df. A linear system of (m) equations in (n) unknowns is a system of the form

$$(*)\left\{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right.$$

where a_{ij}'s and the b_j's are real numbers.

We call this system as $m \times n$ system (linear system).

m = # of equations.

n = # of unknowns.

Ex 1. $\begin{cases} 2x_1 - x_2 = 5 \\ x_1 + 3x_2 = 6 \end{cases}$ is 2×2 linear system.

Ex 2. $\begin{cases} x_1 - x_2 + x_3 = 2 \\ 2x_1 - x_2 + x_3 = 7 \end{cases}$ is 2×3 linear system.

Ex 3. $\begin{cases} x_1 + x_2 = 1 \\ x_1 - x_2 = 2 \\ x_1 = 4 \end{cases}$ is 3×2 linear system.

- Nonlinear system: At least one of the equations in the system are nonlinear

Ex. $\begin{cases} x+y=1 \\ x^2+y=5 \end{cases}$ is nonlinear system.

Rmk. In this course we study only Linear systems.

- By a solution of an $m \times n$ system $(*)$, we mean an ordered n-tuple of numbers $(x_1, x_2, --, x_n)$ that satisfies all the equations of the system.

Ex. $(3,1)$ is a solution of the system

$$\begin{cases} 2x_1 - x_2 = 5 \\ x_1 + 3x_2 = 6 \end{cases}.$$

Sol. $2(3) - 1 = 5 \leftarrow$

$3 + 3(1) = 6 \leftarrow$

2x2 linear systems

$$(**)\quad \begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}$$

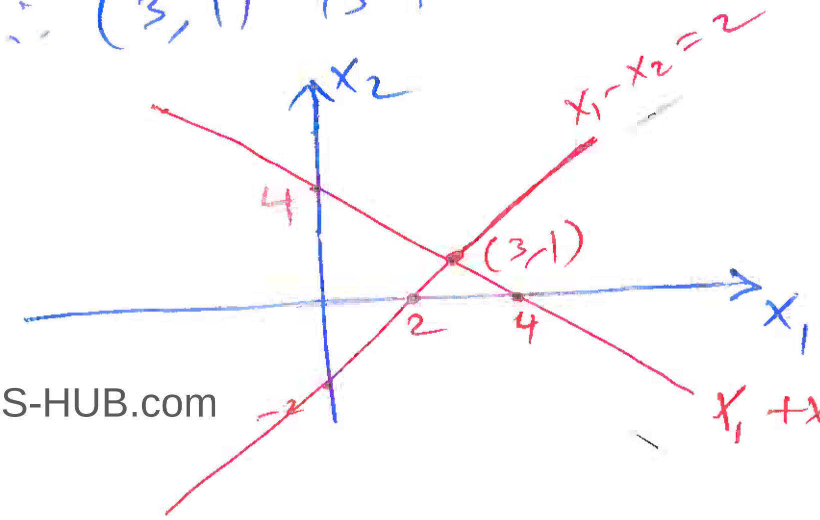
- Each equations in (**) is a <u>line</u> in the plane.
- (x_1, x_2) will be a solution of (**) if and only if it lies on both lines.

Ex. Solve the following systems.

① $\begin{cases} x_1 + x_2 = 4 \\ x_1 - x_2 = 2 \end{cases}$

<u>Sol.</u> Add: $2x_1 = 6 \Rightarrow \boxed{x_1 = 3} \Rightarrow \boxed{x_2 = 1}$

$\therefore$ $(3,1)$ is the solution (unique solution)



② $\begin{cases} x_1 + 2x_2 = 4 & \text{—— (i)} \\ -2x_1 - 4x_2 = 4 & \text{—— (ii)} \end{cases}$

Sol. multiply Eq (i) by 2 : $2x_1 + 4x_2 = 8$ ... (iii)

Add (ii) and (iii) : $0 = 12$ impossible

$\therefore$ the system has no solution.

③ $\begin{cases} 2x_1 - x_2 = 3 \\ -4x_1 + 2x_2 = -6 \end{cases}$

Sol. multiply the first eq. by 2 and add to the second eq : $4x_1 - 2x_2 = 6$

$$\begin{array}{r} 4x_1 - 2x_2 = 6 \\ -4x_1 + 2x_2 = -6 \\ \hline 0 = 0 \end{array}$$

The system has an infinite solution.

How to write it ?

Notice that both equation are the same.

$2x_1 - x_2 = 3 \Rightarrow \boxed{x_2 = 2x_1 - 3}$

let $\boxed{x_1 = t}$, then $\boxed{x_2 = 2t - 3}$, $t \in \mathbb{R}$.

the solution set $= \{(x_1, x_2) = (t, 2t-3) : t \in \mathbb{R}\}$

$\boxed{\text{OR}}$ $2x_1 - x_2 = 3 \Rightarrow \boxed{x_1 = \frac{x_2+3}{2}}$

Let $\boxed{x_2 = r}$, then $\boxed{x_1 = \frac{r+3}{2}}$

The solution set $= \left\{ (x_1, x_2) = \left(\frac{r+3}{2}, r \right) : r \in \mathbb{R} \right\}$.

Rmk. Ingeneral, there are three possibilities for 2x2 linear system:

(i) the lines intersect at a point (Unique solution).

(ii) they are parallel (No solution).

(iii) both equations represent the same line (infinite solution).

Rmk. Ingeneral, there are three possibilities for mxn linear system:

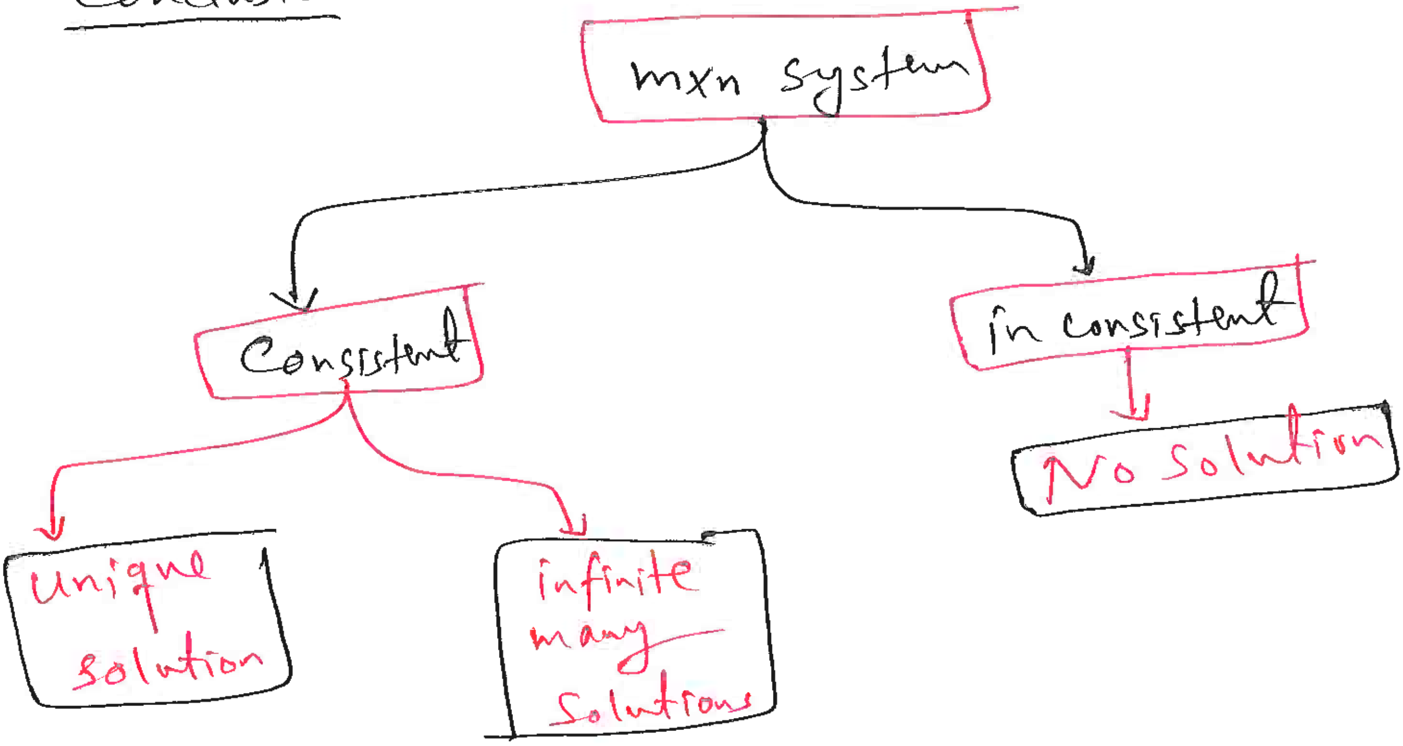
(1) has a unique solution;

(2) has infinitely many solutions.

(3) has no solution.

Df. A linear system is called consistent if it has a solution and it is called inconsistent if it has no solution.

Conclusion



Equivalent systems

Df. Two systems of Equations are called equivalent systems if they have the same variables (unknowns) and the same solution set.

Ex. verify that the following systems are equivalent.

$$(A)\begin{cases} x_1 + x_2 = 2 \\ x_1 - x_2 = 6 \end{cases} \qquad (B)\begin{cases} x_1 + x_2 = 2 \\ 2x_2 = -4 \end{cases}$$

Sol. (i) (A) and (B) have the same variables (x_1 and x_2).

(ii) (A) and (B) have the same solution set $(4,-2)$. ([illegible]).

$\Rightarrow$ The systems (A) and (B) are equivalent.

Df. A linear system is called a square system if $m=n$ and it is called an $n \times n$ linear system.

Df. An $n \times n$ system is said to be in strict triangular form if, in the k^{th} equation the coefficients of the first $k-1$ variables are all zero and the coefficient of x_k is nonzero $(k=1,2,\dots,n)$.

Ex. The system
$$\begin{cases} x_1 + x_2 + x_3 = 8 \\ \qquad 2x_2 + x_3 = 5 \\ \qquad\qquad 3x_3 = 9 \end{cases}$$
is in strict triangular form.

Sol. the system in strict triangular form since the coefficients of the 2^{nd} eq. are $(0,2,1)$

〃 〃 〃 〃 3^d eq. are $(0,0,3)$.

Ex. (H.W) show that the system is in strict triangular form.

$$\begin{cases} x_1 + 2x_2 + 2x_3 + x_4 = 5 & \text{--- (1)} \\ 3x_2 + x_3 - 2x_4 = 1 & \text{--- (2)} \\ -x_3 + 2x_4 = -1 & \text{--- (3)} \\ 4x_4 = 4 & \text{--- (4)} \end{cases}$$

Rmk. (1) the system in strict triangular form is easy to solve and has a unique solution.

(2) we solve the system by backward substitution method as follows.

Ex. Solve the example above by using the back substitution.

Sol. From Eq(4): $4x_4 = 4 \Rightarrow \boxed{x_4 = 1}$

Eq(3): $-x_3 + 2(1) = -1 \Rightarrow \boxed{x_3 = 3}$

Eq(2): $3x_2 + 3 - 2(1) = 1 \Rightarrow \boxed{x_2 = 0}$

Eq(1): $x_1 + 2(0) + 2(3) + 1 = 5 \Rightarrow \boxed{x_1 = -2}$

The solution set $= \{(-2, 0, 3, 1)\}$ (unique).

Question. How to transform a system in strict triangular form ? (if any)

Ans. We use the following elementary row operations.

(I) Interchange two rows (equations).

(II) Multiply a row (eq.) by a nonzero constant.

(III) Replace a row (eq.) by its sum with a multiple of another row (eq.).

Ex. Convert the following system into a strict triangular form and solve it.

$$\begin{cases} -x_2 - x_3 + x_4 = 0 \\ x_1 + x_2 + x_3 + x_4 = 6 \\ 2x_1 + 4x_2 + x_3 - 2x_4 = -1 \\ 3x_1 + x_2 - 2x_3 + 2x_4 = 3 \end{cases}$$

Solution. the augmented matrix of the linear System is $[A:b]$, where A: Coefficients constant
b: constants.

$$[A:b] = \left[\begin{array}{cccc|c} 0 & -1 & -1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 6 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right] \begin{array}{c} R_1 \\ R_2 \\ R_3 \\ R_4 \end{array}$$

$$\xrightarrow[R_1 \text{ and } R_2]{\text{Interchange}} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 2 & 4 & 1 & -2 & -1 \\ 3 & 1 & -2 & 2 & 3 \end{array}\right]$$

$$\xrightarrow{} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 2 & -1 & -4 & -13 \\ 0 & -2 & -5 & -1 & -15 \end{array}\right]$$

$-2R_1+R_3$
$-3R_1+R_4$

$$\xrightarrow{} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -3 & -2 & -13 \\ 0 & 0 & -3 & -3 & -15 \end{array}\right]$$

$2R_2+R_3$
$-2R_2+R_4$

$$\xrightarrow[-R_3+R_4]{} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 6 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & -3 & -2 & -13 \\ 0 & 0 & 0 & -1 & -2 \end{array}\right]$$

The equivalent strict triangular form of that system is

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = 6 \\ \quad -x_2 + x_3 + x_4 = 0 \\ \qquad -3x_3 - 2x_4 = -13 \\ \qquad\qquad -x_4 = -2 \end{cases}$$

By using back substitution, we get the solution set $= \{(-4, 5, 3, 2)\}$

(رجاءً أتحققـــها!).

Rmk (Summary). In general, if an $n \times n$ linear system can be reduced to strictly triangular form, then it will have a unique solution that can be obtained by using back substitution method. However, this method will fail if, at any stage of the reduction process, all the possible choices for a pivot element in a given column are 0. When this happens, the alternative is to reduce the system to certain special echelon or staircase-shaped forms.

We will study these echelon forms in the next section (section 1.2). they will also be used for $m \times n$ systems, where $m \neq n$.

Rmk. ex.
$$\begin{bmatrix} \textcircled{1} & 2 & 3 \\ 0 & \textcircled{4} & 5 \\ 0 & 0 & \textcircled{6} \end{bmatrix}$$
pivotal row

pivot

pivotal column

1.2 Row Echelon Form (REF).

Df. An $m \times n$ matrix is said to be in REF if and only if:

(1) the first nonzero entry in each nonzero row is 1 called the leading one or the pivot 1.

(2) the leading 1 in the k^{th} row is to the right of the leading 1 in the $(k-1)$ row

(3) zero rows are below the nonzero rows.

Ex. $A = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{bmatrix}$ is in REF

$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is not in REF

$C = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 3 & 5 \\ 0 & 0 & 4 \end{bmatrix}$ is not in REF

$$D = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is in REF.

$$E = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$ is in REF.

$$F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ is in REF.

Rmk: (1) Any matrix can be written in REF using the row operations.

(2) the process of using row operations I, II, and III to transform a linear system into one whose augmented matrix is in REF is called

Gaussian Elimination Method.

Ex. Use Gauss Elimination method to solve the following systems.

① $$\begin{cases} x_1 + x_2 = 1 \\ x_1 - x_2 = 3 \\ -2x_1 + 2x_2 = -2 \end{cases}$$

Sol. The augmented matrix is $[A:b]$, i.e.,

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -2 & 2 & -2 \end{array}\right]\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}$$

$$\xrightarrow[\substack{-R_1+R_2 \\ 2R_1+R_3}]{} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 4 & 0 \end{array}\right] \xrightarrow[-\frac{1}{2}R_2]{} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 4 & 0 \end{array}\right]$$

$$\xrightarrow[-4R_2+R_3]{} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 4 \end{array}\right]$$

$$\xrightarrow[\frac{1}{4}R_3]{} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{array}\right]$$

The equivalent system is

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_2 &= -1 \\ 0 &= 1 \quad \text{(impossible)} \end{aligned}$$

$\Rightarrow$ the system is inconsistent (No solution).

② $\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ 2x_1 - x_2 + x_3 = 2 \\ 4x_1 + 3x_2 + 3x_3 = 4 \\ 3x_1 + x_2 + 2x_3 = 3 \end{cases}$

Sol. $\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{array}\right]$

$$\xrightarrow[\substack{-2R_1+R_2 \\ -4R_1+R_3 \\ -3R_1+R_4}]{} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{5}R_2]{} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & \textcircled{1} & \frac{1}{5} & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{5R_2+R_3 \\ 5R_2+R_4}]{} \left[\begin{array}{ccc|c} \textcircled{1} & 2 & 1 & 1 \\ 0 & \textcircled{1} & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The equivalent system is

$$x_1 + 2x_2 + x_3 = 1$$
$$x_2 + \frac{1}{5}x_3 = 0$$

x_1, x_2 leading

x_3 free

let $\boxed{x_3 = t}$

$\Rightarrow \boxed{x_2 = -\frac{1}{5}x_3 = -\frac{1}{5}t}$

$x_1 + 2(-\frac{1}{5}t) + t = 1 \Rightarrow \boxed{x_1 = 1 - \frac{3}{5}t}$

$\therefore$ the solution set is

$$\left\{\left(1-\frac{3}{5}t, -\frac{1}{5}t, t\right) : t \in \mathbb{R}\right\}.$$

③ $\begin{cases} x_1 + 2x_2 + x_3 = 1 \\ 2x_1 + 4x_2 + 2x_3 = 3 \end{cases}$

Sol. the augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{array}\right]$$

$$\xrightarrow[-2R_1+R_2]{} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The system is inconsistent (has nosolution).

④ $\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_1 - x_2 - x_3 = 0 \end{cases}$

Sol. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & -1 & 0 \end{array}\right]$

$$\xrightarrow[-R_1+R_2]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{2}R_2]{} \left[\begin{array}{ccc|c} \textcircled{1} & 1 & 1 & 0 \\ 0 & \textcircled{1} & 1 & 0 \end{array}\right]$$

The equivalent system is

$$x_1 + x_2 + x_3 = 0$$
$$x_2 + x_3 = 0$$

x_1, x_2 leading

x_3 free

Let $\boxed{x_3 = r} \Rightarrow x_2 = -x_3 = -r$

$x_1 + x_2 + x_3 = 0 \Rightarrow x_1 - r + r = 0$

$\boxed{x_1 = 0}$

$\therefore$ the solution set $= \{(0, -r, r) : r \in \mathbb{R}\}$.

Reduced Row Echelon Form (RREF).

Df. An $m \times n$ matrix is said to be in RREF if:

(1) The matrix is in REF.

(2) The first nonzero entry in each row (1's) is the only nonzero entry in its column.

Examples.

$$A = \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$ is in RREF.

$$B = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is in RREF.

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ is NOT in RREF.

$$D = \begin{bmatrix} 1 & 4 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$ is NOT in RREF.

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is in RREF

$$F = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ is not in RREF.

Gauss-Jordan Elimination Method

is the process of using elementary row operations on the augmented matrix $[A:b]$ of the the system $Ax=b$ to transform it into a system in RREF.

Ex. Use Gauss-Jordan reduction to solve

$$\begin{cases} -x_1 + x_2 - x_3 + 3x_4 = 0 \\ 3x_1 + x_2 - x_3 - x_4 = 0 \\ 2x_1 - x_2 - 2x_3 - x_4 = 0 \end{cases}$$

Sol. the augmented matrix is

$$\left[\begin{array}{cccc|c} -1 & 1 & -1 & 3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{-R_1} \left[\begin{array}{cccc|c} \textcircled{1} & -1 & 1 & -3 & 0 \\ 3 & 1 & -1 & -1 & 0 \\ 2 & -1 & -2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[-2R_1+R_3]{-3R_1+R_2} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & 4 & -4 & 8 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right]$$

(24)

$$\xrightarrow[\frac{1}{4}R_2]{} \left[\begin{array}{cccc|c} 1 & -1 & 1 & -3 & 0 \\ 0 & \textcircled{1} & -1 & 2 & 0 \\ 0 & 1 & -4 & 5 & 0 \end{array}\right]$$

$$\xrightarrow[-R_2+R_3]{R_1+R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & -3 & 3 & 0 \end{array}\right]$$

$$\xrightarrow[-\frac{1}{3}R_3]{} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 2 & 0 \\ 0 & 0 & \textcircled{1} & -1 & 0 \end{array}\right]$$

$$\xrightarrow[R_2+R_3]{} \left[\begin{array}{cccc|c} \textcircled{1} & 0 & 0 & -1 & 0 \\ 0 & \textcircled{1} & 0 & 1 & 0 \\ 0 & 0 & \textcircled{1} & -1 & 0 \end{array}\right]$$

x_1, x_2, x_3 are leading variables

x_4 free variable.

let $\boxed{x_4 = t}$

$R_1: \; x_1 - x_4 = 0 \Rightarrow \boxed{x_1 = t}$

$R_2: \; x_2 + x_4 = 0 \Rightarrow \boxed{x_2 = -t}$

$R_3: \; x_3 - x_4 = 0 \Rightarrow \boxed{x_3 = t}$

$\Rightarrow$ the solution set is

$\{(t, -t, t, t) : t \in \mathbb{R}\}$

Overdetermined and underdetermined systems

Df. An $m\times n$ linear system is called underdetermined system if $m<n$, and it is called overdetermined if $m>n$.

Ex ① page 17 is overdetermined system.

Ex ② page 18 // // // .

Ex ③ page 20 // underdetermined system.

Ex ④ page 20 // // // .

Rmk. (i) An underdetermined linear system always has a free variable. So it is either inconsistent or it has infinite solutions. It is not possible to have aunique solution.

(ii) An overdetermined linear system can not tell (All cases possible).

Homogeneous systems

Df: An $m \times n$ lin. system is called homogeneous if all right hand of every equation is zero. that is

$$\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0 \end{array}$$

(i.e $b_1 = b_2 = \cdots = b_n = 0$).

Rmk: (1) A homogenous lin. system is always consistent since $x_1 = x_2 = \cdots = x_n = 0$ is a solution called the zero solution or the trivial solution

(2) A homog. system is either has a unique solution ($x_1 = x_2 = \cdots = x_n = 0$) if it has no free variables or it has infinite solutions if it has a free variable.

(3) An underdetermined homog. linear system always has infinite solutions.

Example. See Ex. page 23.

Ex. Consider the system

$$\begin{cases} x_1 + 2x_2 + x_3 = 0 \\ 2x_1 + 5x_2 + 3x_3 = 0 \\ -x_1 + x_2 + \beta x_3 = 0 \end{cases}$$

(a) Is it possible for the system to be inconsistent? Explain.

(b) for what values of β will the system have infinitely many solution?

Ans. (a) No, since $x_1 = x_2 = x_3 = 0$ is a solution.

(b) The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & 1 & \beta & 0 \end{array}\right]$$

$$\xrightarrow[R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & \beta+1 & 0 \end{array}\right]$$

$$\xrightarrow[-3R_2+R_3]{} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \beta-2 & 0 \end{array}\right]$$

If $\beta \neq 2$, the system has unique solution ($x_1=x_2=x_3=0$)

If $\beta = 2$, " " " infinite solution.

<u>Ex.</u> Consider the linear system

$$\begin{cases} x_1 - x_2 + x_3 = 2 \\ 2x_1 + x_2 - x_3 = 5 \\ x_1 - x_2 + \alpha x_3 = \beta . \end{cases}$$

For what values of α and β does the system have :

(1) unique solution

(2) no solution

(3) infinitely many solutions.

Sol. The augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & 1 & -1 & 5 \\ 1 & -1 & \alpha & \beta \end{array}\right]$$

$$\xrightarrow[-R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 3 & -3 & 1 \\ 0 & 0 & \alpha-1 & \beta-2 \end{array}\right]$$

$$\xrightarrow[\frac{1}{3}R_2]{} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -1 & \frac{1}{3} \\ 0 & 0 & \alpha-1 & \beta-2 \end{array}\right]$$

$$\xrightarrow{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7/3 \\ 0 & 1 & -1 & 1/3 \\ 0 & 0 & \alpha-1 & \beta-2 \end{array}\right]$$

(1) Unique solution if $\alpha \neq 1$, $\beta \in \mathbb{R}$.

(2) No solution if $\alpha = 1$, $\beta \neq 2$

(3) Infinitely many solution if $\alpha = 1$, $\beta = 2$

1.3 Matrix Arithmetic

Df. A matrix is an array of numbers or objects arranged in rows and columns denoted by A, B, C, ---

- A matrix A with m rows and n columns is called an m×n matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & --- & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & --- & a_{mn} \end{bmatrix}_{m\times n}$$

- m×n is the size (order) or the dimension of $A_{m\times n}$

- For similicity, we use the notation

$$A = (a_{ij}), \quad \begin{matrix} i = 1, 2, ---, m \\ j = 1, 2, ---, n \end{matrix}$$

- a_{ij} is called the entry of the matrix A. and it is the entry in the ith row and the jth column.

Ex. $A = \begin{bmatrix} 4 & -8 & 2 \\ 6 & 8 & 10 \end{bmatrix}$ is 2×3 matrix.

Size $= 2\times 3$

$a_{23} = 10$, $a_{22} = 8$

- **Column vector** is an $m\times 1$ matrix

ex. $A = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}_{3\times 1}$.

- **Row vector** is $1\times n$ matrix.

ex. $B = \begin{bmatrix} 1 & 4 & 3 & 6 & 7 \end{bmatrix}_{1\times 5}$.

ex. the solution of the system $\begin{cases} x_1 + x_2 = 3 \\ x_1 - x_2 = 1 \end{cases}$

is $(2,1)$ or $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

- **Euclidean n-space**

$\mathbb{R}^n$: All $n\times 1$ matrices of real entries

ex. $x \in \mathbb{R}^3 \Rightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1, x_2, x_3 \in \mathbb{R}$

$\mathbb{R}^{1\times n}$: All $1\times n$ matrices with real entries

ex. $x \in \mathbb{R}^{1\times 4} \Rightarrow x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}_{1\times 4}$

$\mathbb{R}^{m\times n}$: All $m\times n$ matrices with real entries.

ex. $\mathbb{R}^{3\times 2} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} : a_{ij} \in \mathbb{R} \begin{array}{l} \\ i=1,2,3 \\ j=1,2 \end{array} \right\}$

- If A is $m\times n$ matrix, then the row vectors of A are $\vec{a_i} = (a_{i1}, a_{i2}, ---, a_{in})$, $i=1,2,---,n$.

$$\Rightarrow A = \begin{pmatrix} \vec{a_1} \\ \vec{a_2} \\ \vdots \\ \vec{a_n} \end{pmatrix}$$

and the column vectors of A are

$$a_j = \begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}, \quad j=1,2,---,n.$$

$$\Rightarrow A = (a_1, a_2, ----, a_n).$$

Ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & -1 \end{bmatrix}_{2\times 3}$

The row vectors of A are

$\vec{a_1} = (1,2,3)$, $\vec{a_2} = (0,4,-1)$

The column vectors of A are

$$a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

Df. (Equality of matrices) Two matrices A and B are equal iff they have the same size and $a_{ij} = b_{ij}, \forall i, j$.

Ex. let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. then,

$A \neq B$ since $a_{ij} \neq b_{ij}$

Ex. let $A = \begin{bmatrix} 1 & 3 \\ 2x+1 & 3y^2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 \\ 3 & 9 \end{bmatrix}$.

If $A = B$, then find x and y.

Sol. A and B have the same size 2x2.

$\Rightarrow$ $2x+1 = 3$. and $3y^2 = 9$

$\boxed{x = 1}$, $\boxed{y = \pm\sqrt{3}}$

Operations on matrices

Scalar multiplication

Df: let A be an $m \times n$ matrix, α be scalar (real or complex), then

$$\boxed{\alpha A = (\alpha a_{ij}), \forall i,j}.$$

Ex. $A = \begin{bmatrix} -2 & 1 \\ 0 & 5 \end{bmatrix}$. Find $6A$.

Sol. $6A = \begin{bmatrix} 6\times-2 & 6\times1 \\ 6\times0 & 6\times5 \end{bmatrix} = \begin{bmatrix} -12 & 6 \\ 0 & 30 \end{bmatrix}$

Matrix Addition and Subtraction

Df. If $A = (a_{ij})$ and $B = (b_{ij})$ are both $m \times n$ matrices, then

$$\boxed{A \pm B = (a_{ij} \pm b_{ij}), \forall i,j}$$

Ex. let $A = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} -1 & 5 \\ 6 & 7 \end{bmatrix}$. Find the following.

① $A - C$ ② $A + B$ ③ $2B - 3A$

Sol. ① $A - C =$ undefined

② $A + B = \begin{bmatrix} 5 & 2 & 5 \\ 4 & 5 & 6 \end{bmatrix}$

③ $2B - 3A = \begin{bmatrix} 4 & 0 & 8 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 9 & 6 & 3 \\ 12 & 15 & 18 \end{bmatrix}$

$= \begin{bmatrix} -5 & -6 & 5 \\ -12 & -15 & -18 \end{bmatrix}$.

- zero matrix O is amatrix whose entries are all zero.

ex. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times 2}$, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}_{3\times 1}$, $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}_{1\times 4}$

are all zero matrices.

Properties of addition & scalar multiplication

Thm. let A, B be an $m\times n$ matrices, $\alpha, \beta \in \mathbb{R}$ then

① $\alpha(A+B) = \alpha A + \alpha B$

② $(\alpha\beta)A = \alpha(\beta A)$

③ $A+B=B+A$.

④ $A+(B+C)=(A+B)+C$

⑤ $A+O=O+A=A$.

⑥ $A-A=A+(-A)=O$

$-A$ is called the <u>additive inverse</u> of A.

Matrix Multiplication and linear systems

Df: let A be $m\times n$, B an $n\times k$ matrices. Then $AB=C$, where $c_{ij}=\sum_{k=1}^{n} a_{ik}b_{kj}$

Ex. let $A=\begin{bmatrix}1 & 3\\ 6 & -1\end{bmatrix}$, $B=\begin{bmatrix}1 & 5 & 2\\ 0 & 1 & -3\end{bmatrix}$, then

$$AB=\underset{2\times2}{\begin{bmatrix}1 & 3\\ 6 & -1\end{bmatrix}}\underset{2\times3}{\begin{bmatrix}1 & 5 & 2\\ 0 & 1 & -3\end{bmatrix}}$$

$$=\underset{2\times3}{\begin{bmatrix}1\times1+3\times0 & 1\times5+3\times1 & 1\times2+3\times-3\\ 6\times1+-1\times0 & 6\times5+-1\times1 & 6\times2+-1\times-3\end{bmatrix}}$$

$$=\underset{2\times3}{\begin{bmatrix}1 & 8 & -7\\ 6 & 29 & 15\end{bmatrix}}$$

BA is undefined

Ingeneral, $AB\neq BA$

ex.
$$\begin{bmatrix}1 & 2\end{bmatrix}_{1\times2}\begin{bmatrix}2\\1\end{bmatrix}_{2\times1} = \begin{bmatrix}1\times2 + 2\times1\end{bmatrix}_{1\times1}$$
$$= \begin{bmatrix}4\end{bmatrix}_{1\times1}$$

$$\begin{bmatrix}2\\1\end{bmatrix}_{2\times1}\begin{bmatrix}1 & 2\end{bmatrix}_{1\times2} = \begin{bmatrix}2\times1 & 2\times2\\1\times1 & 1\times2\end{bmatrix}_{2\times2} = \begin{bmatrix}2 & 4\\1 & 2\end{bmatrix}_{2\times2}$$

Linear Systems and matrices

A linear system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\\ &\vdots\\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

Can be written in matrix multiplication as

$$Ax = b, \text{ where}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m\times n}, \quad x = \begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_n \end{pmatrix}_{n\times1}, \quad b = \begin{pmatrix} b_1\\ \vdots\\ b_m \end{pmatrix}_{m\times1}$$

$A_{m\times n}$ coefficients matrix.

$x\in\mathbb{R}^n$ (unknowns), $b\in\mathbb{R}^m$ (constants)(knowns)

Ex. Write the system in a matrix form:

$$4x_1 + 2x_2 + x_3 = 1$$

$$5x_1 + 3x_2 + 7x_3 = 2$$

Sol.
$$\underbrace{\begin{bmatrix} 4 & 2 & 1 \\ 5 & 3 & 7 \end{bmatrix}}_{A_{2\times3}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X_{3\times1}} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{b_{2\times1}}$$

Also, we can write the linear system $Ax=b$

as $\quad a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$

or $\quad \begin{bmatrix} \vec{a_1}x \\ \vec{a_2}x \\ \vdots \\ \vec{a_m}x \end{bmatrix} = b$, where a_i columns

$\vec{a_i}$ rows of A.

Ex. In the last example

$$b=\begin{bmatrix}1\\2\end{bmatrix}=Ax=\begin{bmatrix} 4 & 2 & 1 \\ 5 & 3 & 7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= x_1\begin{bmatrix}4\\5\end{bmatrix} + x_2\begin{bmatrix}2\\3\end{bmatrix} + x_3\begin{bmatrix}1\\7\end{bmatrix}$$

$$= x_1a_1 + x_2a_2 + x_3a_3$$

Also, $b=\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix} [4 \;\; 2 \;\; 1]x \\ [5 \;\; 3 \;\; 7]x \end{bmatrix} = \begin{bmatrix} \vec{a_1}x \\ \vec{a_2}x \end{bmatrix}$

Rmk: ① A vector X_0 is a solution of a linear system $Ax=b$ iff $AX_0=b$.

② If X_1, X_2 are solutions of $Ax=b$, then $\alpha X_1+\beta X_2$ is a solution of $Ax=b$ iff $\alpha+\beta=1$.

③ X_1, X_2 are solutions of a homog. linear system $Ax=0$, then $\alpha X_1+\beta X_2$ is a solution of $Ax=0$, $\forall \alpha, \beta \in \mathbb{R}$.

Df (linear combination)

If $a_1, a_2, \ldots, a_n$ are vectors in $\mathbb{R}^m$ and $c_1, c_2, \ldots, c_n$ are scalars, then the sum $\boxed{c_1 a_1 + c_2 a_2 + \cdots + c_n a_n}$ is said to be a linear combination of the vectors $a_1, a_2, \ldots, a_n$.

Ex. Is $b = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$ a linear combination of

$a_1 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, $a_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$?

Ans. Yes, since $b = 1a_1 + 0a_2$.

Ex. Is $b = \begin{pmatrix} 2 \\ 24 \end{pmatrix}$ a linear combination of

$a_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $a_2 = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$?

Ans. $b = c_1 a_1 + c_2 a_2$

$$\begin{pmatrix} 2 \\ 24 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

$\Rightarrow \boxed{c_1 = 2}$, $2c_1 + 5c_2 = 24 \Rightarrow \boxed{c_2 = 4}$

$\therefore b = 2a_1 + 4a_2$

$\therefore b$ is a linear combination of a_1 and a_2.

Ex. Is $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ a lin. combination of

$a_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $a_2 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$?

Ans. $b = c_1 a_1 + c_2 a_2 \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$$\Rightarrow -2(c_1 + 2c_2 = 1$$

$$\begin{array}{r} 2c_1 + 4c_2 = 1 \\ \hline 0 = -1 \end{array}$$ impossible (inconsistent)

$\Rightarrow$ b is not a lin. combination of a_1 & a_2.

Thm (Consistency of the linear system)

A linear system $Ax=b$ is consistent if and only if b is a linear combination of the columns of A (i.e, $b = x_1a_1 + x_2a_2 + \cdots + x_na_n$).

Proof. ($\Rightarrow$) Suppose that $Ax=b$ is consistent, So there exist real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $A\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = b$. So,

$$\alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n = b$$

and so b is a linear combination of the columns of A.

($\Leftarrow$) Conversely,

Suppose that b is a linear combination of the columns of A, so there exist real numbers $\alpha_1, \alpha_2, ---, \alpha_n$ such that

$$b = \alpha_1 a_1 + \alpha_2 a_2 + ---+ \alpha_n a_n$$

$$\Rightarrow b = A\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{pmatrix}.$$ So, $\begin{pmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_n\end{pmatrix}$ is a solution of $Ax=b$. that is, $Ax=b$ is consistent ▨

Rmk. the proof of the last theorem shows that if b is a linear combination of the columns of A, then the coefficients of the column of A is a solution of $Ax=b$

Ex. let $\boxed{A_{3\times3}}$, $\boxed{Ax=b}$, $\boxed{b = 4a_1 - 6a_2 + 3a_3}$, then $\begin{pmatrix}4\\-6\\3\end{pmatrix}$ is a solution of $Ax=b$. Then the system is consistent (the system has a unique solution or infinite solutions).

Ex. Q12) $A_{3\times4}$. If $b = a_1 + a_2 + a_3 + a_4$. Is the system $Ax=b$ consistent?

If yes, what can conclude about the number of solutions?

<u>Ans.</u> the system is <u>consistent</u> since $\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$ is a solution of $Ax=b$.

Since the system is underdetermined and consistent, so it has infinite solutions.

<u>Ex.</u> If $A_{3\times3}$ and $a_3 = a_1 - a_2$, then $Ax=0$ has infinitely many solutions.

<u>Sol.</u> $a_1 - a_2 - a_3 = 0 \Rightarrow (1,-1,-1)$ is a solution of $Ax=0$.

Since the homog. system $Ax=0$ has a none zero solution, it has infinite solutions.

<u>Ex.</u> Let $A_{4\times3}$ with $a_1 = a_2$. If $b = a_1+a_2+a_3$, then the system $Ax=b$ has infinitely many solutions.

.(Exercise)

The Transpose of a matrix

Df. the transpose of an $m \times n$ matrix A is defined by $A^T = (a_{ij})^T = (a_{ji}) = B_{n\times m}$.

Ex. (a) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, then $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$.

(b) If $B = [1 \quad 2 \quad 3]$, $B^T = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

(c) If $C = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $C^T = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = C$.

(d) If $D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $D^T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = -D$

Df. An $n \times n$ matrix A is said to be symmetric if $A^T = A$.

Ex. (c) above

Df. An $n \times n$ matrix A is said to be skew-symmetric if $A^T = -A$.

Ex. (d) above.

Properties of the transpose

(1) $(A^T)^T = A$

(2) $(A \pm B)^T = A^T \pm B^T$

(3) $(\alpha A)^T = \alpha A^T$, α scalar.

(4) $(AB)^T = B^T A^T$

(5) If $A_{n\times n}$, $B_{n\times n}$ are symmetric, then $A+B$ is symmetric

(6) If $A_{n\times n}$ is symmetric, then αA is symmetric, $\forall \alpha \in \mathbb{R}$.

Ex. Prove the following. (H.W).

(1) Let A and B be symmetric matrices. Then $H = AB - BA$ is skew symmetric.

(2) If A is an $m \times n$ matrix. Then $A^T A$ and $A A^T$ both symmetric.

(3) If A is symmetric and skew-symmetric, then A must be zero matrix.

1.4 Matrix Algebra

Theorem For any scalars α and β and any matrices A, B, and C, the following valid.

(1) $A+B=B+A$

(2) $(A+B)+C = A+(B+C)$

(3) $(AB)C = A(BC)$.

(4) $A(B+C) = AB+AC$.

(5) $(A+B)C = AC+BC$.

(6) $(\alpha\beta)A = \alpha(\beta A)$.

(7) $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

(8) $(\alpha+\beta)A = \alpha A+\beta A$.

(9) $\alpha(A+B) = \alpha A+\alpha B$.

(10) $A^n = \underbrace{A \cdot A \cdots A}_{n\text{-times}}$

Examples

Ex 1. If $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B=\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$, and $C=\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$. verify that

(i) $A(BC) = (AB)C$

(ii) $A(B+C) = AB + AC$.

Sol. (i) $A(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \left(\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right)$

$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 16 & 11 \end{bmatrix}$

$(AB)C = \left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} -4 & 5 \\ -6 & 11 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ 16 & 11 \end{bmatrix}$

$\therefore A(BC) = \begin{bmatrix} 6 & 5 \\ 16 & 11 \end{bmatrix} = (AB)C$.

(ii) $A(B+C) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \left(\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \right)$

$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 5 & 15 \end{bmatrix}$

$AB + AC = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} -4 & 5 \\ -6 & 11 \end{bmatrix} + \begin{bmatrix} 5 & 2 \\ 11 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 5 & 15 \end{bmatrix}$

therefore, $A(B+C) = AB + AC$.

Ex. If $A=\begin{bmatrix}1&1\\1&1\end{bmatrix}$, find A^{2020}.

Sol. $A^2=\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}1&1\\1&1\end{bmatrix}=\begin{bmatrix}2&2\\2&2\end{bmatrix}$

$A^3=AAA=AA^2=\begin{bmatrix}1&1\\1&1\end{bmatrix}\begin{bmatrix}2&2\\2&2\end{bmatrix}=\begin{bmatrix}4&4\\4&4\end{bmatrix}$

$A^4=AAAA=A^3A=\begin{bmatrix}4&4\\4&4\end{bmatrix}\begin{bmatrix}1&1\\1&1\end{bmatrix}=\begin{bmatrix}8&8\\8&8\end{bmatrix}$

$\vdots$

$A^n=\begin{bmatrix}2^{n-1}&2^{n-1}\\2^{n-1}&2^{n-1}\end{bmatrix}$

$\therefore A^{2020}=\begin{bmatrix}2^{2019}&2^{2019}\\2^{2019}&2^{2019}\end{bmatrix}$

The identity matrix

Df. The $n\times n$ identity matrix is the matrix

$I=(\delta_{ij})$, where $\delta_{ij}=\begin{cases}1 & \text{if } i=j\\0 & \text{if } i\neq j\end{cases}$

Ex. For $n=3$, $I_3=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

For $n=2$, $I_2=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

Rmk. If B is any $m\times n$ matrix and
C " " $n\times r$ matrix, then

$$BI = B \quad \text{and} \quad IC = C, \text{ where } I_{n\times n}.$$

Matrix Inversion

Df. An $n\times n$ matrix A is said to be nonsingular or invertible if there exists a matrix B such that $\boxed{AB = BA = I.}$

- The matrix B is called the inverse or multiplicative inverse of A denoted by A^{-1}.
- If A^{-1} does not exist, then A has no inverse (A is called singular or not invertible)

Ex. Let $A = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{pmatrix}$.

Verify that A and B are inverses of each other.

Sol. $AB = \begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} -\frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & -\frac{1}{5} \end{pmatrix} = \begin{pmatrix} -\frac{2}{10} + \frac{12}{10} & \frac{4}{5} - \frac{4}{5} \\ -\frac{3}{10} + \frac{3}{10} & \frac{6}{5} - \frac{1}{5} \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$$BA=\begin{pmatrix} \frac{-1}{10} & \frac{2}{5} \\ \frac{3}{10} & \frac{-1}{5} \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 3 & 1 \end{pmatrix}=\begin{pmatrix} \frac{-1}{5}+\frac{6}{5} & \frac{-4}{10}+\frac{2}{5} \\ \frac{6}{10}-\frac{3}{5} & \frac{12}{10}-\frac{1}{5} \end{pmatrix}$$

$$=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=I$$

$\therefore$ $AB=BA=I$.

Ex. Show that $A=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ has no inverse (singular matrix).

Pf. If B is any 2×2 matrix, then

$$BA=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} b_{11} & 0 \\ b_{21} & 0 \end{pmatrix}\neq I .$$

Rmk. ONLY square matrices have multiplicative inverse. We should not use the terms singular or nonsingular to nonsquare matrices.

Ex. (How to find A^{-1}, if $A_{2\times2}$ matrix).

Let $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Show that if

$\boxed{\alpha=ad-bc\neq0}$, then $\boxed{A^{-1}=\frac{1}{\alpha}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$

Rmk. If $\alpha=0$, then A^{-1} DNE (A is singular).

Proof. $$AA^{-1} = \frac{1}{\alpha}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \frac{1}{\alpha}\begin{pmatrix} ad-bc & -ab+ba \\ dc-dc & -cb+ad \end{pmatrix}$$

$$= \frac{1}{\alpha}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A^{-1}A = \frac{1}{\alpha}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \frac{1}{\alpha}\begin{pmatrix} ad-bc & bd-bd \\ -ac+ac & -bc+ad \end{pmatrix}$$

$$= \frac{1}{\alpha}\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$\blacksquare$

$$\therefore A^{-1}A = AA^{-1} = I.$$

Ex. If $A = \begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}$, find A^{-1} (if any).

Sol. $\alpha = (4)(2) - (3)(2) = 8 - 6 = 2 \neq 0$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{2} \\ -1 & 2 \end{bmatrix}$$

Ex. $A = \begin{bmatrix} 3 & 2 \\ 9 & 6 \end{bmatrix}$ $\quad \alpha = (3)(6) - (2)(9)$

$= 18 - 18 = 0$

$\Rightarrow A$ has no inverse (Singular).

Theorem If A and B are nonsingular $n \times n$ matrices, then AB is also nonsingular and $\boxed{(AB)^{-1} = B^{-1}A^{-1}}$.

Proof. $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I$

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$ ∎

$\therefore (AB)^{-1} = B^{-1}A^{-1}$

Algebraic Rules for Inverses

(1) the inverse of A if exists is unique.

(2) $(A^{-1})^{-1} = A$

(3) $(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$, α is scalar.

(4) If A is invertible, then A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$.

(5) $[(AB)^T]^{-1} = (A^{-1})^T(B^{-1})^T$

(6) If $A_1, A_2, ---, A_k$ are nonsingular, then $A_1A_2----A_k$ is nonsingular and $(A_1A_2---A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}-----A_2^{-1}A_1^{-1}$.

Exercise Prove or disprove.

(1) If A and B are invertible matrices, then $A+B$ is also invertible (nonsingular).

(2) The sum of singular matrices is also singular

False. Counterexample $A = \overset{\text{sing.}}{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}$, $B = \overset{\text{sing}}{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}$, but $A+B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ nonsingula

(3) $A^2 - B^2 = (A-B)(A+B)$, where A and B are matrices

// // //

(4) $(A+B)^2 = A^2 + 2AB + B^2$

(5) If $AB = 0$, then $A = 0$ or $B = 0$.

(6) If $A^2 = 0$, then $A = 0$

(7) If $AB = AC$, then $B = C$

(8) If $A^2 = A$, then $A = O$ or $A = I$.

(9) If $A_{n\times n}$ matrix, such that $A^2 = A$, then $I+A$ is nonsingular and $(I+A)^{-1} = I - \frac{1}{2}A$.

Ans: (True)

proof: $(I+A)(I-\frac{1}{2}A) = I^2 - \frac{1}{2}IA + AI - \frac{1}{2}A^2$

$= I - \frac{1}{2}A + A - \frac{1}{2}A$

$= I + O = I$.

$(I-\frac{1}{2}A)(I+A) = I$ (أفتح تفضله).

$\therefore (I+A)^{-1} = I - \frac{1}{2}A$. ▤

(10) Let A be an $n\times n$ matrix. then if $A^2 = O$, then $I - A$ is nonsingular and $(I-A)^{-1} = I + A$.

(11) If A and B are $n\times n$ matrices. Then if $AB = A$ and $B \neq I$, then A must be singular

(True)

Proof. If A were nonsingular, then A^{-1} exists.

thus $A^{-1}(AB) = A^{-1}A \Rightarrow IB = I \Rightarrow B = I$

which is a contradiction. therefore, A must be Singular ▤

1.5 Elementary Matrices

Df. A matrix E is called an elementary matrix if it is obtained from the identity I_n by performing exactly one row operations.

There are three types:

Type I: E is obtained from I_n by interchanging any two rows of I_n: $\boxed{\text{Notation } E^{(1)}}$.

Ex. $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an elementary matrix of type I.

Type II: E is obtained from I_n by multiplying any row of I_n by a nonzero constant.

$\boxed{\text{Notation } E^{(2)}.}$

Ex. $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2019 \end{bmatrix}$ is $E^{(2)}$

Type III: E is obtained from I_n by adding a multiple of one row of I_n to another row of I_n.

Ex. $C = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{-3R_3+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

$\therefore$ C is $E^{(3)}$.

Rmk. ① Similarly, we have column elementary matrices by performing similar operations on the columns of I_n. But we study the the row elementary matrices.

Rmk. ② Multiplying a matrix A from left by an elementary matrix is the same as performing a row operation on A of the same type.

Ex. let $A_{3\times3}$ and $E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is $E^{(1)}$

$$EA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

of the same type $E^{(1)}$.

Rmk ③. Multiplying a matrix A from right by a column elementary matrix is the same as performing a column operation on A of the same type.

ex. let $A_{3\times3}$ and $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is $E^{(2)}$

$$AE = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & 2a_{12} & a_{13} \\ a_{21} & 2a_{22} & a_{23} \\ a_{31} & 2a_{32} & a_{33} \end{bmatrix} \text{ is } E^{(2)}$$

Theorem. If E is an elementary matrix, then E is nonsingular (invertible) and E^{-1} is an elementary of the same type.

ex. $E = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ is $E^{(2)}$.

$$E^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \text{ is } E^{(2)}.$$

Df. A matrix B is a row equivalent to a matrix A if there exists a finite sequence of elementary matrices $E_1, E_2, \cdots, E_n$ such that $B = E_k E_{k-1} \cdots E_1 A$.

In another words, B is row equivalent to A if B can be obtained from A by a finite of row operations.

Ex. If $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 3 \\ 2 & 2 & 6 \end{bmatrix}$, and

$$C = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -1 & -3 \\ 2 & 2 & 6 \end{bmatrix}.$$

(a) Find an elementary matrix E such that $EA = B$ (i.e, B is row equivalent to A)

Ans. $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

(b) Find an elementary matrix F such that $FB = C$ (i.e C is row equivalent to B).

Ans. $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

(c) Is C row equivalent to A ?

Yes, since $C = FB$ (part (b))
$= FEA$ (part (a))
$\therefore C = (\underbrace{FE}_{\text{elem.}})A$

Rmk. (i) If A is row equivalent to B,
then B 〃 〃 〃 〃 A.

(ii) If A is row equivalent to B and B is row equivalent to C, then A is row equivalent to C.

Proof. Suppose that A is row equivalent to B
then $A = (E_k E_{k-1} \cdots E_1)B$, where
$E_1, E_2, \ldots, E_k$ elementary matrices

$$\Rightarrow (E_k E_{k-1} \cdots E_1)^{-1} A = (E_k \cdots E_1)^{-1} (E_k \cdots E_1) B$$

$$(E_1^{-1} E_2^{-1} \cdots E_k^{-1}) A = IB = B .$$

$$\therefore \; B = \underbrace{E_1^{-1} E_2^{-1} \cdots E_k^{-1}}_{\text{elementary (see Thm p.57).}} A$$

i.e, B is row equivalent to A.

(ii) By assumption, we have

$$A = (E_k E_{k-1} \cdots E_1) B$$

and $$B = (F_k F_{k-1} \cdots F_1) C ,$$ where

$E_1, \ldots, E_k, F_1, \ldots, F_k$ are elementary matrices. Then,

$$A = (E_k E_{k-1} \cdots E_1 F_k F_{k-1} \cdots F_1) C$$

i.e, A is row equivalent to C ■

<u>Theorem</u>. Let A be an $n \times n$ matrix. Then the following statements are <u>equivalent</u>:

(a) A is nonsingular.

(b) $Ax = 0$ has only the trivial solution ($x = 0$ the zero solution).

(c) A is row equivalent to I_n.

<u>Proof</u>. $\boxed{(a) \Rightarrow (b)}$: Suppose that A is nonsingular and y is a solution of $Ax = 0$, i.e, $Ay = 0$. multiply both sides by A^{-1} from left, we get $A^{-1}(Ay) = A^{-1}0$. So $Iy = 0$, i.e, $y = 0$

$\boxed{(b) \Rightarrow (c)}$: Suppose that $Ax = 0$ has only zero solution ~~of~~ Suppose that A is not row equivalent to I, so the RREF of A has a free variable and so $Ax = 0$ has infinitely many solutions which is a contradiction.

$\boxed{(c) \Rightarrow (a):}$ suppose that A is row equivalent to I_n, so there exist a finite sequence of elementary matrices $E_1, E_2, \dots, E_k$ such that $(E_k E_{k-1} \cdots E_1) I = A$. So,

$$A = E_k E_{k-1} \cdots E_1 \quad \text{and so}$$

$$A^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

$\therefore$ A is invertible (nonsingular)

Corollary. The system $Ax = b$, where $A_{n\times n}$, has a unique solution if and only if A is nonsingular. (the solution is $\boxed{x = A^{-1}b}$)

Ex. (True) or (False)

If A is a 4×4 matrix and $a_1 + a_2 = a_3 + 2a_4$ then A must be singular.

(True). $a_1 + a_2 - a_3 - 2a_4 = 0 \Rightarrow (1, 1, -1, -2)^T$ is a solution of $Ax = 0$

$\Rightarrow$ $Ax = 0$ has infinitely many solution.

$\Rightarrow$ A is singular (thm).

Rmk. the above thm gives a strategy to find the inverse of a square matrix if it exist as follows.

Ex. If $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{bmatrix}$. Find A^{-1} (if any)

Solution. **Strategy:** $[A \mid I_3] \xrightarrow[\text{operations}]{\text{row}} [I_3 \mid A^{-1}]$

$$[A \mid I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -2 & -3 & -2 & 0 & 1 \end{array}\right]$$

$-2R_1+R_3$

$-R_2+R_1$

$$\xrightarrow{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right]$$

$2R_2+R_3$

$$\xrightarrow[-2R_3+R_2]{} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 4 & -3 & -2 \\ 0 & 0 & 1 & -2 & 2 & 1 \end{array}\right]$$

$$\underbrace{}_{I_3} \qquad \underbrace{}_{A^{-1}}$$

$$\therefore A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 4 & -3 & -2 \\ -2 & 2 & 1 \end{bmatrix}$$

Rmk. If in the process of performing row operations on $(A \mid I_n)$ one row of A reduced to a zero row, then A^{-1} does not exist. (A singular).

Ex. If $A = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 2 & -1 \\ -2 & 2 & -6 \end{bmatrix}$. Find A^{-1} (if any).

Sol.

$$[A \mid I] = \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 1 & 2 & -1 & 0 & 1 & 0 \\ -2 & 2 & -6 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[2R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 1 & 0 & 0 \\ 0 & 3 & -4 & -1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \end{array}\right]$$

$\therefore$ A has no inverse.

- Solving the square system $Ax = b$ using the inverse of A (if exists): $\boxed{x = A^{-1}b}$

Ex. Solve the system

$$\begin{cases} x_1 + x_2 + 2x_3 = -2 \\ \phantom{x_1 +{}} x_2 + 2x_3 = 3 \\ 2x_1 \phantom{{}+ x_2} + x_3 = 0 \end{cases}$$

Sol. $$\underbrace{\begin{bmatrix}1&1&2\\0&1&2\\2&0&1\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}}_{x}=\underbrace{\begin{bmatrix}-2\\3\\0\end{bmatrix}}_{b}$$

$$\Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1&1&2\\0&1&2\\2&0&1\end{bmatrix}^{-1}\cdot\begin{bmatrix}-2\\3\\0\end{bmatrix}$$

$$=\begin{bmatrix}1&-1&0\\4&-3&-2\\-2&2&1\end{bmatrix}\begin{bmatrix}-2\\3\\0\end{bmatrix} \quad (\text{see } A^{-1}\text{ p. }63)$$

$$=\begin{bmatrix}-2-3\\-8-9\\4+6\end{bmatrix}=\begin{bmatrix}-5\\-17\\10\end{bmatrix}$$

$\Rightarrow x_1=-5,\ x_2=-17,\ x_3=10$ (unique soluti

Diagonal and Triangular matrices.

Df. Let $A_{n\times n}$ matrix.

(1) If $a_{ij}=0$, for $i>j$, then A is called upper triangular.

(2) If $a_{ij}=0$, $\forall i<j$, then A is called lower triangular

(3) A is said to be triangular if it is either upper or lower triangular.

(4) If $a_{ij} = 0, \forall i \neq j$, then A is called diagonal matrix.

Ex. $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 0 & 0 \\ 0 & 4 & 3 \end{bmatrix}$ lower triangula

upper triangular

$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Diagonal

diagonal and triangular.

LU factorization (Triangular Factorization)

Df. If a matrix A is reduced into an upper triangular matrix using row operations of type III only, then A has a triangular factorization $A = LU$, where U is upper triangular and L is unit lower triangular.

Rmk. Not every matrix has an LU factorization.

Ex. Compute the LU factorization of the matrix $A = \begin{bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{bmatrix}$.

Solution. $A = \begin{bmatrix} -2 & 1 & 2 \\ 4 & 1 & -2 \\ -6 & -3 & 4 \end{bmatrix}$

$$\xrightarrow[\substack{2R_1+R_2 \\ -3R_1+R_3}]{} \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & -6 & -2 \end{bmatrix}$$

$$\xrightarrow[2R_2+R_3]{} \begin{bmatrix} -2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 2 \end{bmatrix} = U$$

- $L = ??$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$$

$$E_1 = I(2R_1+R_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (2R_1+R_2)$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_2 = I(-3R_1+R_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} -3R_1+R_3$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E_3 = I(2R_2+R_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} 2R_2+R_3$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

So, $E_3 E_2 E_1 A = U$

$$\Rightarrow A = (E_3 E_2 E_1)^{-1} U$$

$$A = \underbrace{E_1^{-1} E_2^{-1} E_3^{-1}}_{L} U$$

That is $L = E_1^{-1} E_2^{-1} E_3^{-1}$

To find E_1^{-1}:

$$[E_1 : I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\underbrace{\quad}_{I} \qquad \underbrace{\quad}_{E_1^{-1}}$$

$$\therefore E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly, you can find E_2^{-1}, E_3^{-1}:

$$E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}, \quad E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$L = E_1^{-1} E_2^{-1} E_3^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -2 & 1 \end{bmatrix}$$

Ex. Find the LU factorization of

$$A = \begin{bmatrix} 0 & -1 & 3 \\ 2 & 4 & -1 \\ -2 & 2 & -4 \end{bmatrix} \text{ if it exists.}$$

Solution. A has no LU factorization.
(why ?!).

Ex. Find the LU decomposition of the matrix $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$.

Solution. $U = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$$

(رجاءً أقنع نفسك)

Ex. (H.W). (True) or (False):

(1) If A has an LU-factorization, then A is nonsingular iff L is nonsingular.

(2) If A has an LU-factorization, then A is nonsingular iff U is nonsingular.

(3) If A has an LU-factorization, then A is row equivalent to U.

The End of chapter 1

Chapter 2. Determinants

2.1 The determinant of a matrix

Df. If A is an $n\times n$ matrix, then the determinant of A is denoted by $\boxed{\det(A)}$ or $\boxed{|A|}$

Case 1. 1x1 Matrices

If $A=(a)$ is 1x1 matrix, then we define

$$\boxed{\det(A)=a}$$

Ex. If $A=(-2020)$, then $|A|=-2020$.

Case 2. 2x2 Matrices

If $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$ is a 2x2 matrix, then we define

$$\boxed{\det(A)=a_{11}a_{22}-a_{12}a_{21}}$$

Ex. If $A=\begin{bmatrix} -2 & 1 \\ 4 & 5\end{bmatrix}$, then $|A|=(-2)(5)-(1)(4)$
$=-10-4=-14$.

Thm. An $n\times n$ matrix A is nonsingular iff $|A|\neq 0$

Ex. Is $A=\begin{bmatrix} 3 & 2 \\ 9 & 6\end{bmatrix}$ invertible (nonsingular)?

An. No, since $|A|=(3)(6)-(2)(9)=0$ (A is singular

Ex. If $A = \begin{bmatrix} 2-\lambda & 4 \\ 3 & 3-\lambda \end{bmatrix}$ is singular. Find the values of λ.

Sol. A is singular $\Rightarrow |A| = 0$

$$\Rightarrow (2-\lambda)(3-\lambda) - 12 = 0$$

$$\lambda^2 - 5\lambda - 6 = 0$$

$$(\lambda - 6)(\lambda + 1) = 0$$

$$\therefore \boxed{\lambda = 6} \text{ or } \boxed{\lambda = -1}$$

Cofactor Method

Df. Let A be an $n \times n$ matrix and let M_{ij} be an $(n-1)\times(n-1)$ matrix obtained from A by deleting the row and column containing a_{ij}. Then,

$$\boxed{m_{ij} = \text{the minor of } a_{ij} = \det(M_{ij})}$$

$$\boxed{A_{ij} = \text{the cofactor of } a_{ij} = (-1)^{i+j} \det(M_{ij})}$$

Ex. If $A = \begin{bmatrix} 2 & 5 & 4 \\ 3 & 1 & 2 \\ 5 & 4 & 6 \end{bmatrix}$, find m_{13}, A_{32}.

Sol. $m_{13} = \begin{vmatrix} 3 & 1 \\ 5 & 4 \end{vmatrix} = 12-5 = 7$, $A_{32} = (-1)^{3+2} m_{32}$

$$= -\begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} = 8$$

Df. let A be an $n \times n$ matrix. Then we define the $\det(A)$ by

$$\det(A) = \begin{cases} a_{11}, & \text{if } n=1 \\ a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}, & \text{if } n \geq 2. \end{cases}$$

This is called the expansion of the $\det(A)$ along the first row of A.

Ex. If $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & -2 & 3 \\ 2 & 3 & 2 \end{bmatrix}$. Find $|A|$.

Sol. $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$

$$= 3(-1)^{1+1}m_{11} + 2(-1)^{1+2}m_{12} + 4(-1)^{1+3}m_{13}$$

$$= 3m_{11} - 2m_{12} + 4m_{13}$$

$$= 3\begin{vmatrix} -2 & 3 \\ 3 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}$$

$$= 3(-4-9) - 2(2-6) + 4(3+4)$$

$$= -39 + 8 + 28 = -3.$$

Notice that since $|A| = -3 \neq 0$, then A is nonsingual (invertible).

Ex. Find $\begin{vmatrix} 1 & 0 & 0 & 0 \\ 5 & 6 & 0 & 0 \\ 7 & 8 & 2 & 0 \\ -2 & 1 & 5 & 6 \end{vmatrix}$

Sol. Let A be the matrix,

$$|A| = a_{11}A_{11} + \cancelto{0}{a_{12}A_{12}} + \cancelto{0}{a_{13}A_{13}} + \cancelto{0}{a_{14}A_{14}}$$

$$= 1(-1)^2 \begin{vmatrix} 6 & 0 & 0 \\ 8 & 2 & 0 \\ 1 & 5 & 6 \end{vmatrix}$$

$$= (1)(1)(6)(-1)^2 \begin{vmatrix} 2 & 0 \\ 5 & 6 \end{vmatrix} = (6)(12) = 72$$

Notice that $|A| = (1)(6)(2)(6)$ is the product of the entries in the main diagonal. This is the case in general for any triangular matrix.

Ex. $A = \begin{bmatrix} 3 & 2 & 4 \\ 1 & -2 & 3 \\ 2 & 3 & 2 \end{bmatrix}$ find $|A|$ along the second column.

Sol. $|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$

$$= 2(-1)^3 \begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} - 2(-1)^4 \begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} + 3(-1)^5 \begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix}$$

$$= -2(2-6) - 2(6-8) - 3(9-4) = -3$$

Thm. If A is $n\times n$ matrix, $n \geq 2$, then $\det(A)$ can be expressed as a cofactor expansion using any row or column of A.

$$\det(A) = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$
$$= a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$$

for $i = 1, 2, \cdots, n$ and $j = 1, 2, \cdots, n$.

Ex. $\begin{vmatrix} 0 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & 0 & 1 & 3 \end{vmatrix}$ (we use the first column)

$$= \cancelto{0}{a_{11}} A_{11} + \cancelto{0}{a_{21}} A_{21} + \cancelto{0}{a_{31}} A_{31} + a_{41}A_{41}$$

$$= -2\begin{vmatrix} 2 & 3 & 0 \\ 4 & 5 & 0 \\ 1 & 0 & 3 \end{vmatrix} = -2\cdot 3\cdot \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 12$$

Also, see example p. 75.

Thm. If A is an $n\times n$ matrix, then

$$\boxed{\det(A^T) = \det(A).}$$

Proof. the proof is by induction on n. See the textbook p. 109.

Thm. If A is an $n \times n$ triangular matrix, then the det(A) equats the product of the diagonal elements of A. (See ex. p.75).

Thm. Let A be an $n \times n$ matrix.

(i) If A has a row or column consistining entirely of zeros, then $\det(A) = 0$

(ii) If A has two identical rows or two identical columns, then $\det(A) =$

Proof. (See Exercises 9 and 10 p.110 textbook.).

Ex.

$$\begin{vmatrix} 1 & 2 & 4 & 6 \\ 5 & -1 & 0 & 4 \\ 6 & 7 & -9 & 10 \\ 1 & 2 & 4 & 6 \end{vmatrix} = 0$$

Ex.

$$\begin{vmatrix} 1 & 4 & 5 & 6 & 9 \\ -2 & 1 & 0 & 1 & 2 \\ -4 & 6 & 7 & 9 & 1 \\ 1 & 5 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

2.2 Properties of Determinants

lemma. Let A be an $n\times n$ matrix. Then

$$a_{i1}A_{j1}+a_{i2}A_{j2}+\cdots+a_{in}A_{jn}=\begin{cases}0, & i\neq j\\ |A|, & i=j\end{cases}$$

proof. If $i=j$, then

$$a_{i1}A_{j1}+a_{i2}A_{j2}+\cdots+a_{in}A_{jn}=a_{i1}A_{i1}+\cdots+a_{in}A_{in}$$
$$=|A|$$

If $i\neq j$, let A^* be the matrix obtained from A by replacing the jth row by the ith row, i.e,

$$A^*=\left[\begin{array}{cccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{array}\right]\begin{array}{l} \\ \\ j\text{th row} \\ \\ i\text{th row} \\ \\ \\ \end{array}$$

Since two rows of A^* are the same, so $|A^*|=0$.

So, $0 = \det(A^*) = a_{i1} A^*_{j1} + a_{i2} A^*_{j2} + \cdots + a_{in} A^*_{jn}$

$= a_{i1} A_{j1} + a_{i2} A_{j2} + \cdots + a_{in} A_{jn}$ ▤

Row operations

Thm. let A be asquare matrix and B is obtained from A by only one row operation. then

1) Type I (B is obtained by interchaning two rows of A). Then $|B| = -|A|$.

ex. $A = \begin{bmatrix} 2 & 4 \\ 6 & 5 \end{bmatrix}$, $|A| = 10 - 24 = -14$.

$B = \begin{bmatrix} 6 & 5 \\ 2 & 4 \end{bmatrix}$, $|B| = 24 - 10 = +14$.

notice that $|B| = -|A|$.

2) Type II. B is obtained from A by multiplying one row only of A by a nonzero constant $(\alpha \neq 0)$. then

$|B| = \alpha |A|$.

ex. $A = \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \Rightarrow |A| = 5-8 = -3.$

$B = \begin{bmatrix} 2 & 4 \\ 4 & 5 \end{bmatrix} \Rightarrow |B| = 10-16 = -6 = 2(-3)$
$= 2|A|.$

Type III. B is obtained from A by adding a multiple of one row of A to another row of A. Then $|B| = |A|$.

ex. $A = \begin{bmatrix} 1 & 4 \\ 5 & -5 \end{bmatrix}, \ |A| = -5-20 = -25$

$B \underset{-5R_1+R_2}{=} \begin{bmatrix} 1 & 4 \\ 0 & -25 \end{bmatrix}, \ |B| = -25 = |A|.$

thm. Let E be an elementary matrix. Then

$$\det(E) = \begin{cases} -1 & \text{if } E \text{ is of type I} \quad E^{(1)} \\ \alpha \neq 0 & \text{if } E \text{ is of type II} \quad E^{(2)} \\ 1 & \text{if } E \text{ is of type III} \quad E^{(3)} \end{cases}$$

thm. let E be an elementary matrix, and let A be a matrix of the same size of E. Then $|EA| = |E||A|$.

Thm. Let $E_1, \dots, E_k$ be elementary matrices. Then $|E_1 \cdots E_k| = |E_1| \cdots |E_k|$.

Proof. Use math. induction.

Thm. An $n \times n$ matrix A is Singular iff $\det(A) = 0$.

OR. An $n \times n$ matrix A is nonsingular iff $|A| \neq 0$

proof. ($\Rightarrow$) let A be nonsingular. So, A is row equivalent to I_n. That is, there exist elementary matrices $E_1, E_2, \dots, E_k$ such that

$$A = E_1 \cdots E_k I_n .$$

So $|A| = |E_1| |E_2| \cdots |E_k| \neq 0$.

($\Leftarrow$) Conversely, suppose that $|A| \neq 0$.

Then the matrix A can be changed to RREF with a finite number of row operations. That is, there exist elementary matrices $E_1, \cdots, E$ and a matrix U in RREF such that

$$E_k E_{k-1} \cdots E_1 U = A$$

. Since $|A| \neq 0$, So, $|U| \neq 0$, since all E_i's are invertible, and $|A| = |E_1||E_2| \cdots |E_k||U|$. So, $U = I_n$, and so A is invertible. ▤

Thm. If A, B are $n \times n$ matrices, then

$$\det(AB) = \det(A)\det(B).$$

Proof. If A is singular, then $|A| = 0$ and so AB is singular, and therefore,

$$|AB| = 0 = |A||B|.$$

If A is nonsingular, then A is row equivale to I_n. That is, $A = E_1 \cdots E_k I_n = E_1 \cdots E_k$

Thus, $|AB| = |E_1 E_2 \cdots E_k B|$

$= |E_1||E_2| \cdots |E_k||B| = |A||B|$ ▤

Ex. If $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 5$. Find

$$\begin{vmatrix} 2a & 2b & 2c \\ d & e & f \\ g+a & h+b & i+c \end{vmatrix}$$

Sol. $\begin{vmatrix} 2a & 2b & 2c \\ d & e & f \\ g+a & h+b & i+c \end{vmatrix}$

$$= 2\begin{vmatrix} a & b & c \\ d & e & f \\ g+a & h+b & i+c \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 2(5) = 10.$$

$-R_1+R_3$

Ex. Prove or disprove.

① $|A+B| = |A|+|B|$, where $+$ is defined.

② $|A^n| = |A|^n$, $n = 0, 1, 2, \ldots$

③ $|kA| = k^n |A|$, where $A_{n\times n}$, $k\in\mathbb{R}$.

④ If A is nonsingular, then $\det(A^{-1}) = \frac{1}{\det(A)}$.

⑤ If $A^2 = A$, then $|A| = 0$ or $|A| = 1$.

⑥ If $A^T A = I$, then $|A| = \pm 1$.

⑦ If $A_{n\times n}$ is skew-symmetric and n is odd, then A must be singular.

⑧ If $A_{n\times n}$ is skew-symmetric and n is even, then A must be nonsingular.

⑨ Let $A_{n\times n}$, $B_{n\times n}$. Then AB is nonsingular iff A and B are both nonsingular.

⑩ If A, B, and C are 3×3 matrices, $|A|=9$, $|B|=2$, $|C|=3$. Then,

$$\left|4C^TBA^{-1}\right| = \frac{128}{3}.$$

2.3 Additional Topics and Applications

. The Adjoint of a Matrix

Df. Let A be nxn matrix. The adjoint of A is defined as

$$adj(A) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}^T, \text{ where}$$

$$A_{ij} = (-1)^{i+j}|M_{ij}|$$

Ex. Find adj(A) if $A = \begin{bmatrix} -1 & 3 \\ 4 & 6 \end{bmatrix}$

Sol. $A_{11} = +6$, $A_{12} = -\det(4) = -4$

$A_{21} = -\det(3) = -3$, $A_{22} = +\det(-1) = -1$

$$\therefore adj(A) = \begin{bmatrix} 6 & -4 \\ -3 & -1 \end{bmatrix}^T = \begin{bmatrix} 6 & -3 \\ -4 & -1 \end{bmatrix}$$

Thm. Let A be nxn matrix, then

$$A \, adj(A) = |A| I_n$$

Proof: the ij^{th} entry of $A\,\mathrm{adj}(A)$ is

$$A\,\mathrm{adj}(A) = a_{i1}A_{j1} + \cdots + a_{in}A_{jn} = \begin{cases} |A|, & i=j \\ 0, & i \neq j \end{cases}$$

$$= |A|\, I_n \quad ▤$$

Thm: let A be $n\times n$ nonsingular matrix. then $A^{-1} = \frac{1}{|A|}\mathrm{adj}(A)$.

Proof: Since A is nonsingular, then A^{-1} exists. From last thm, we know, $A\,\mathrm{adj}(A) = |A|\, I_n$. multiply both sides by A^{-1} from left:

$$A^{-1}A\,\mathrm{adj}(A) = |A|\, A^{-1} I_n = |A|\, A^{-1}$$

$\mathrm{adj}(A) = |A|A^{-1}$ and so, since $|A| \neq 0$,

$$\Rightarrow \quad A^{-1} = \frac{1}{|A|}\mathrm{adj}(A). \quad ▤$$

Ex: Let $A = \begin{bmatrix} 2 & 1 & 2 \\ 3 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$. compute $\mathrm{adj}(A)$ and A^{-1}.

Sol. $|A| = 2\begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} - 1\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} + 2\begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix}$

$= 2(6-4) - 1(9-2) + 2(6-2) = 5 \neq 0.$

$\therefore A^{-1}$ exists.

$$\operatorname{adj}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 1 & 3 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & -7 & 4 \\ 1 & 4 & -3 \\ -2 & 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} \operatorname{adj}(A) = \frac{1}{5} \begin{bmatrix} 2 & 1 & -2 \\ -7 & 4 & 2 \\ 4 & -3 & 1 \end{bmatrix}.$$

Cramer's Rule

Thm. Let A be a nonsingular $n \times n$ matrix, and let $b \in \mathbb{R}^n$. Let A_i be the matrix obtained by replacing the ith column of A by b. If x is the unique solution of $Ax = b$, then

$$\boxed{x_i = \frac{\det(A_i)}{\det(A)}, \quad i = 1, 2, \ldots, n}$$

Proof. Since $x = A^{-1} b = \frac{1}{|A|} \operatorname{adj}(A) b$

it follows that

$$x_i = \frac{b_1 A_{1i} + b_2 A_{2i} + \cdots + b_n A_{ni}}{|A|}$$

$$= \frac{\det(A_i)}{\det(A)}.$$

∎

Ex. Use Cramer's rule to solve

$$x_1 + 2x_2 + x_3 = 5$$

$$2x_1 + 2x_2 + x_3 = 6$$

$$x_1 + 2x_2 + 3x_3 = 9$$

Solution

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 9 \end{bmatrix}$$

$$|A| = -4, \quad |A_1| = \begin{vmatrix} 5 & 2 & 1 \\ 6 & 2 & 1 \\ 9 & 2 & 3 \end{vmatrix} = -4$$

$$|A_2| = \begin{vmatrix} 1 & 5 & 1 \\ 2 & 6 & 1 \\ 1 & 9 & 3 \end{vmatrix} = -4, \quad |A_3| = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 2 & 6 \\ 1 & 2 & 9 \end{vmatrix} = -8$$

Therefore,

$$x_1 = \frac{|A_1|}{|A|} = \frac{-4}{-4} = 1, \quad x_2 = \frac{|A_2|}{|A|} = \frac{-4}{-4} = 1$$

$$x_3 = \frac{|A_3|}{|A|} = \frac{-8}{-4} = 2.$$

Section 2-3 (selected exercises).

Q8) Let A be a nonsingular $n \times n$ matrix with $n > 1$. Show that $|\operatorname{adj} A| = |A|^{n-1}$.

Proof: If A is nonsingular, then $|A| \neq 0$ and hence $\operatorname{adj} A = |A| A^{-1}$ is also non singular.

$$\Rightarrow |\operatorname{adj} A| = \Big| \underset{\text{constant}}{\boxed{|A|}} A^{-1} \Big| = |A|^n \, |A^{-1}|$$

$$= |A|^n \cdot \frac{1}{|A|} = |A|^{n-1} \quad \#.$$

Q10) Show that if A is nonsingular, then $\operatorname{adj} A$ is nonsingular and

$$\boxed{(\operatorname{adj} A)^{-1} = |A^{-1}| A = \operatorname{adj} A^{-1}} \quad (*)$$

Proof: If A is nonsingular, then $|A| \neq 0$ and hence $\operatorname{adj} A = |A| A^{-1}$ is also nonsingular.

To prove $(*)$, we have

$$(\operatorname{adj} A)^{-1} = \Big(\underbrace{|A|}_{\text{ثابت}} A^{-1} \Big)^{-1} = \frac{1}{|A|} (A^{-1})^{-1} = \frac{1}{|A|} A$$

$$= |A^{-1}| A \quad — ①$$

and $$\operatorname{adj} A^{-1} = |A^{-1}| (A^{-1})^{-1} = |A^{-1}| A \quad — ②$$

hence ① & ② gives $(*)$. #.

Example let A be $n\times n$ matrix (nonsingular), n

then $\boxed{\operatorname{adj}(\operatorname{adj} A) = |A|^{n-2} A}$.

Proof: Since A is nonsingular, then $|A| \neq 0$.

and hence $\operatorname{adj} A = |A| A^{-1}$

$$\Rightarrow \operatorname{adj}(\operatorname{adj} A) = \operatorname{adj}\left(\boxed{|A| A^{-1}}\right)$$

انظرلها كأنها B

$$= \left| \underbrace{|A|}_{\text{ثابت}} A^{-1} \right| \left(\underbrace{|A|}_{\text{ثابت}} A^{-1} \right)^{-1}$$

$$= |A|^n |A^{-1}| \frac{1}{|A|} (A^{-1})^{-1}$$

$$= |A|^n \cdot \frac{1}{|A|} \cdot \frac{1}{|A|} \cdot A$$

$$= |A|^{n-2} A.$$

Q12 Show that if $|A| = 1$, then

$$\operatorname{adj}(\operatorname{adj} A) = A.$$

Proof: This question is a special case of the above example. We proved that

$$\operatorname{adj}(\operatorname{adj} A) = |A|^{n-2} A$$

If $|A| = 1$, then $\operatorname{adj}(\operatorname{adj} A) = (1)^{n-2} A = A$. ▬

Rmk: Another special case if $A_{2\times 2}$, then $\operatorname{adj}(\operatorname{adj} A) = A$.

(إقتنعوا فيها)

Chapter 3. vector spaces

3.1 Definition and Examples

Df. A vector space V is a set of elements together with the operations of addition and scalar multiplication such that the following axioms are satisfied:

C1: If $x \in V$ and α is scalar, then $\alpha x \in V$ "closed under scalar multiplication".

C2: If $x, y \in V$, then $x+y \in V$ "closed under addition".

A1: $x+y = y+x, \quad \forall x, y \in V$

A2: $(x+y)+z = x+(y+z), \quad \forall x, y, z \in V$.

A3: $\exists$ an element $0 \in V$ such that

$$x+0 = 0+x = x, \quad \forall x \in V.$$

A4: $\forall x \in V, \ \exists \ -x \in V$ such that

$$x + -x = 0$$

A5: $\alpha(x+y) = \alpha x + \alpha y$, for each α scalar and $x, y \in V$.

A6: $(\alpha+\beta)x = \alpha x + \beta x$, for each α, β scalars and $x \in V$.

A7: $(\alpha\beta)x = \alpha(\beta x)$, $\forall x \in V$, α, β scalars.

A8: $1x = x$, $\forall x \in V$.

Notation $(V, +, \cdot)$.

Examples

Ex① $(\mathbb{R}, +, \cdot)$: $\mathbb{R}$ with usual addition and multiplication is a vector space.

Ex② $V = \mathbb{R}^2$ with usual $+$ and $\cdot$ is a vector space where

$$(a,b) + (c,d) = (a+c, b+d).$$

$$\alpha(a,b) = (\alpha a, \alpha b)$$

<u>Pf ②</u>. Let α be scalar and $x=(a,b)$, $y=(c,d)$, $z=(e,f) \in \mathbb{R}^2$. then

C1. $\alpha x = \alpha(a,b) = (\alpha a, \alpha b) \in \mathbb{R}^2$.

$\therefore$ $V=\mathbb{R}^2$ is closed under scalar multiplication.

C2: $x+y = (a,b)+(c,d)$
$= (a+c, b+d) \in \mathbb{R}^2$

$\therefore$ $V=\mathbb{R}^2$ is closed under addition.

A1: $x+y = (a,b)+(c,d)$
$= (a+c, b+d) = (c+a, d+b)$
$= (c,d)+(a,b) = y+x$.

A2: $x+(y+z) = (a,b)+\left[(c,d)+(e,f)\right]$
$= (a,b)+(c+e, d+f)$
$= (a+c+e, b+d+f)$
$= \left[(a,b)+(c,d)\right]+(e,f)$
$= (x+y)+z$

A3: $x + 0 = (a,b) + (0,0) = (a+0, b+0)$
$= (a,b) = x, \quad \forall x \in V$

$\therefore 0 = (0,0) \in \mathbb{R}^2$

A4: $\forall x = (a,b) \in \mathbb{R}^2, \ \exists \ -x = (-a,-b) \in \mathbb{R}^2$
such that $(a,b) + (-a,-b) = (0,0)$.

A5: $\alpha(x+y) = \alpha[(a,b) + (c,d)]$
$= \alpha(a+c, b+d) = (\alpha a + \alpha c, \alpha b + \alpha d)$
$= (\alpha a, \alpha b) + (\alpha c, \alpha d)$
$= \alpha(a,b) + \alpha(c,d)$
$= \alpha x + \alpha y$

A6: $(\alpha+\beta)x = (\alpha+\beta)(a,b)$
$= ((\alpha+\beta)a, (\alpha+\beta)b)$
$= (\alpha a + \beta a, \alpha b + \beta b)$
$= (\alpha a, \alpha b) + (\beta a, \beta b)$
$= \alpha(a,b) + \beta(a,b) = \alpha x + \beta x.$

A7: $(\alpha\beta)x = (\alpha\beta)(a,b)$

$$= ((\alpha\beta)a, (\alpha\beta)b)$$

$$= (\alpha(\beta a), \alpha(\beta b))$$

$$= \alpha(\beta a, \beta b)$$

$$= \alpha[\beta(a,b)] = \alpha(\beta x).$$

A8: $1x = 1(a,b) = (1a, 1b) = (a,b) = x$

$$\forall x = (a,b) \in \mathbb{R}^2.$$

$\therefore (\mathbb{R}^2, +, \cdot)$ is a vector space.

Rmk. $(\mathbb{R}^n, +, \cdot)$ " " "

Ex③. $M_{m\times n} = \mathbb{R}^{m\times n}$ is the set of all mxn matrices with real entries under addition and scalar multiplication is a vector space.

Ex④. the set of all real valued functions under + and · :

$$\begin{cases} (f+g)(x) = f(x) + g(x) \\ (\alpha f)(x) = \alpha f(x) \end{cases}$$

is a vector space.

The zero polynomial is $O(x) = 0$ of degree zero.

Ex 5. $C[a,b] := \{ f : [a,b] \longrightarrow \mathbb{R} : f$ is continuous on $[a,b] \}$ under addition and scalar multiplication of functions:

$(f+g)(x) = f(x) + g(x)$, $(\alpha f)(x) = \alpha f(x)$ is a vector space.

Ex 6. $C^n[a,b] = \{ f : [a,b] \longrightarrow \mathbb{R} : f^{(n)}$ is continuous on $[a,b] \}$ under addition and scalar multiplication of functions:

$(f+g)(x) = f(x) + g(x)$, $(\alpha f)(x) = \alpha f(x)$

is a vector space.

Ex.7. $P_n = \{ f(x) = a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$ $a_0, a_1, \cdots, a_{n-1} \in \mathbb{R} \}$ under addition and scalar multiplication of functions:

$$(f+g)(x) = f(x) + g(x) \,,\ (\alpha f)(x) = \alpha f(x)$$

is a vector space.

Ex. $P_3 = \{ f(x) = ax^2 + bx + c,\ a, b, c \in \mathbb{R} \}$

$P_2 = \{ f(x) = ax + b \,,\ a, b \in \mathbb{R} \}.$

Ex 8. $\mathbb{Q} = \{ x = \frac{a}{b} : \ a, b$ are integers, $b \neq 0 \}$ is not a vector space.

Pf. A1: $x = \frac{2}{5} \in \mathbb{Q}$, $\alpha = \sqrt{2}$ scalar but

$$\alpha x = \sqrt{2} \cdot \frac{2}{5} = \frac{2\sqrt{2}}{5} \notin \mathbb{Q}.$$

Ex 9. $\mathbb{Z} = \{ 0, \pm 1, \pm 2, \cdots \}$ is not a vector space under usual addition and scalar multiplication.

Ex. 10. $V = \{ f(x) : \deg(f) = 3 \}$ is not a vector space under usual $+, \cdot$ of functions.

Pf: $1 - x^3, 1 + x^3 \in V$ but $1 - x^3 + 1 + x^3 = 2 \notin V$

Ex. 11 $V = \{ (1, y) : y \in \mathbb{R} \}$ under usual addition and scalar multiplication is not a vector space.

Ex. 12. $V = \{ (0, y, 0) : y \in \mathbb{R} \}$ under usual addition + scalar multiplication is a vector space.

Theorem. Let V be a vector space. Then

(i) $0v = \vec{0}$, $\forall v \in V$.

(ii) If $x + y = \vec{0}$, then $y = -x$.

(iii) $-1 \cdot v = -v$, $\forall v \in V$.

Proof: (i) $0 = 0 + 0$, so $(0+0)v = 0v$.

thus $0v + 0v = 0v$. add to both sides

$-0v$. So, $0v + 0v + -0v = 0v + -0v$.

$\Rightarrow$ $0v + \vec{0} = \vec{0}$. Hence $0v = \vec{0}$.

(ii) Add $-x$ to both sides of $x + y = \vec{0}$.

So, $-x + x + y = -x + \vec{0}$. thus, $\vec{0} + y = -x + \vec{0}$

$\Rightarrow y = -x$.

(iii) $0 = 1 + -1$, So $(1 + -1)v = 0v = \vec{0}$

Thus, $1v + -1v = \vec{0}$, so $v + -1v = \vec{0}$

$\Rightarrow -v + v + -1v = -v + \vec{0} = -v$.

$\Rightarrow$ $\vec{0} + -1v = -v$. Thus, $-1v = -v$

3.2 Subspace and spanning sets

Df. A nonempty subset S of a vector space V is called a subspace of V iff the following holds.

1) $x+y \in S$, $\forall x, y \in S$.

2) $\alpha x \in S$, $\forall x \in S, \forall \alpha \in \mathbb{R}$.

Thm. Let S be a subspace of a vector space V. Then $\vec{0} \in S$.

proof. Since S is a subspace of V, then $S \neq \Phi$. Let $x \in S$, so $0x = \vec{0} \in S$ ▤

Rmk. Let S be a subset of a Vector space V. If $\vec{0} \notin S$, then S is not a subspace of V.

Examples.

Ex 1. $S = \{ (a,b)^T : a+b=1 , a,b \in \mathbb{R} \}$ is not a subspace of $\mathbb{R}^2$.

pf. $(0,0) \notin S$. $\boxed{\text{OR}}$ $(1,0), (0,1) \in S$ but $(1,0)+(0,1)=(1,1) \notin S$.

Ex 2. $S = \{ (1,b) : b \in \mathbb{R} \}$, $V = \mathbb{R}^2$

S is not a subspace.

Ex 3. $S = \{ A_{n\times n} : \det(A) \neq 0 \}$, $V = \{A_{n\times n}\}$

S is not a subspace

Ex 4. $S = \{ A_{n\times n} : |A| = 0 \}$, $V = \{ A_{n\times n} \}$

S is not a subspace.

Ex 5. $S = \{(0, y, z) : y, z \in \mathbb{R}\}$, $V = \mathbb{R}^3$

S is a subspace.

Sol. (i) $S \neq \phi$ since $(0,0,0) \in S$.

(ii) Let $(0, x, y)$, $(0, a, b) \in S$. Then

$(0,x,y) + (0,a,b) = (0, x+a, y+b) \in S$.

(iii) Let $\alpha \in \mathbb{R}$, $(0, y, z) \in S$. Then

$\alpha(0, y, z) = (0, \alpha y, \alpha z) \in S$.

Ex. 6. Let S be the set of all symmetric $m \times n$ matrices. That is, $S = \{A_{n \times n} : A^T = A\}$, $V = \{A_{n \times n}\}$. Then S is a subspace of V

Ex.7 Let $S = \{ A_{2\times 2} : a_{12} = -a_{21} \}$.

Then S is a subspace of $\mathbb{R}^{2\times 2}$.

Pf. $S = \left\{ \begin{bmatrix} a & b \\ -b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$

(i) $S \neq \phi$ since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$.

(ii) Let $A = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}$, $B = \begin{bmatrix} d & e \\ -e & f \end{bmatrix} \in S$.

Then $A + B = \begin{bmatrix} a+d & b+e \\ -b-e & c+f \end{bmatrix} \in S$.

(iii) Let $\alpha \in \mathbb{R}$, $A = \begin{bmatrix} a & b \\ -b & c \end{bmatrix} \in S$. Then

$$\alpha A = \begin{bmatrix} \alpha a & \alpha b \\ -\alpha b & \alpha c \end{bmatrix} \in S.$$

$\therefore$ S is a subspace of $\mathbb{R}^{2\times 2}$.

Ex.8 $S = \{$The set of all polynomials $p(x) \in P_4$ such that $p(0) = 0\}$ is a subspace of P_4.

Ex 9: The set of all polynomials in P_4 having at least one real roots is not a subspace of P_4.

Ex 10. $S = \{A_{n\times n} : A \text{ is triangular}\}$ is not a subspace of $\{A_{n\times n}\}$.

Theorem. Let S and T be subspales of a vector space V. then

(i) $S \cap T$ is a subspace.

(ii) $S \cup T$ is not allways a subspace of V.

(iii) $S + T = \{x+y : x \in S, y \in T\}$ is a subspace of V.

Proof. (i). Since $0 \in S$, $0 \in T$, then

$$0 \in S \cap T \Rightarrow S \cap T \neq \phi$$

- let $x, y \in S \cap T$, then $x, y \in S$ and $x, y \in T$

$\Rightarrow x+y \in S$ and $x+y \in T$

$\Rightarrow x+y \in S \cap T$.

- let $\alpha \in \mathbb{R}$ and $x \in S \cap T$.

Since $x \in S \cap T$, then $x \in S$ and $x \in T$.

Since S and T are subspaces, it follows

$\alpha x \in S$ and $\alpha x \in T$

$\Rightarrow \alpha x \in S \cap T$.

(ii) let $S = \{ (x,0) : x \in \mathbb{R} \}$

$T = \{ (0,y) : y \in \mathbb{R} \}$.

Notice that S and T are subspaces of $\mathbb{R}^2$

but $S \cup T = \{ (x,y) : x$ or y is zero $\}$

is not subspace, for example,

$(0,1), (1,0) \in S \cup T$, but $(0,1) + (1,0) = (1,1) \notin S \cup T$

(iii) Exercise

The Null space of a matrix

Df. let A be $m\times n$ matrix. The null space of A is $N(A) = \{ x \in \mathbb{R}^n : Ax = 0 \}$.

Ex. If $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}_{2\times 4}$. Find $N(A)$.

Sol. $N(A) = \{ x \in \mathbb{R}^4 : Ax = 0 \}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[-2R_1+R_2]{} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right] \xrightarrow[-R_2]{} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2+R_1} \left[\begin{array}{cccc|c} \textcircled{1} & 0 & -1 & 1 & 0 \\ 0 & \textcircled{1} & 2 & -1 & 0 \end{array}\right]$$

x_1, x_2 are leading variables

$\boxed{x_3 = t}$, $\boxed{x_4 = r}$ are free variables

The equivalent system is

$$x_1 - x_3 + x_4 = 0 \Rightarrow \boxed{x_1 = t - r}$$

$$x_2 + 2x_3 - x_4 = 0 \Rightarrow \boxed{x_2 = -2t + r}$$

$\therefore N(A) = \{ (t-r, -2t+r, t, r)^T : t, r \in \mathbb{R} \}$.

Theorem: let A be $m \times n$ matrix. Then $N(A)$ is a subspace of $\mathbb{R}^n$.

proof: (i) since $A0 = 0$, then $0 \in N(A)$

$\therefore N(A) \neq \phi$.

(ii) let $x, y \in N(A)$. Then $Ax = 0$ and $Ay = 0$, So $A(x+y) = Ax + Ay$

$= 0 + 0 = 0$

$\Rightarrow x + y \in N(A)$.

(iii) let $x \in N(A)$, $\alpha \in \mathbb{R}$. Then $Ax = 0$, so $A(\alpha x) = \alpha Ax = \alpha \cdot 0 = 0$

$\Rightarrow \alpha x \in N(A)$ ▤

Linear Combinations

Df. Let V be a vector space and let $v_1, v_2, ---, v_k \in V$, $c_1, c_2, --, c_k \in \mathbb{R}$. Then a vector $c_1 v_1 + c_2 v_2 + ---+ c_k v_k$ is called a linear combination of $v_1, v_2, ---, v_k$.

The set of all linear combinations of $V_1, V_2, \dots, V_k$ is called the Span of $V_1, V_2, \dots, V_k$ denoted by $\text{Span}(V_1, \dots, V_k)$

ex. Is $V = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ a linear combination of $V_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $V_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$?

Ans. Let $V = c_1 V_1 + c_2 V_2$. Then

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \begin{array}{l} c_1 + c_2 = 2 \\ c_2 = 3 \\ \hline \end{array}$$

$$\Rightarrow c_1 = -1, \; c_2 = 3$$

$\therefore$ $V = -V_1 + 3V_2$

i.e, V is a linear combination of V_1 and V_2 OR $V \in \text{Span}(V_1, V_2)$.

Ex. Is $f(x) = x \in \text{Span}(1, 3x)$?

Sol. Let $f(x) = x = c_1 \cdot 1 + c_2 \cdot 3x$

$$\Rightarrow \boxed{c_1 = 0}, \; \boxed{c_2 = \frac{1}{3}}$$

$\therefore$ $f(x) = x = 0(1) + \frac{1}{3} \cdot (3x)$

$\therefore$ f is a linear combination of 1 and $3x$

$\Rightarrow f(x) = x \in \text{Span}(1, 3x)$.

Ex. In $\mathbb{R}^3$, find $\text{Span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$.

Sol. $\text{Span}\left[\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right] = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} : \begin{pmatrix}x\\y\\z\end{pmatrix} = \alpha\begin{pmatrix}1\\0\\0\end{pmatrix} + \beta($

$\Rightarrow \alpha = x, \ \beta = y.$
$z = 0$

$\therefore \text{Span}\left[\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right] = \left\{\begin{pmatrix}x\\y\\0\end{pmatrix} : x, y \in \mathbb{R}\right\}.$

Ex. In $\mathbb{R}^2$, find $\text{span}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right)$

Sol. $\text{Span}\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right) = \{ V \in \mathbb{R}^2 : V = \alpha_1\begin{pmatrix}1\\0\end{pmatrix} + \alpha_2\begin{pmatrix}0\\1\end{pmatrix}$

$= \left\{\begin{pmatrix}x\\y\end{pmatrix} : \begin{pmatrix}x\\y\end{pmatrix} = \alpha_1\begin{pmatrix}1\\0\end{pmatrix} + \alpha_2($

$\alpha_1 = x, \ \alpha_2 = y$

$= \left\{\begin{pmatrix}x\\y\end{pmatrix} : x, y \in \mathbb{R}\right\}$

$= \mathbb{R}^2.$

Thm: let V be a vector space and let $v_1, v_2, ---, v_k \in V$. Then $\text{Span}(v_1, ---, v_k)$ is a subspace of V.

Proof: (i) Since $\vec{0} = 0 \cdot V_1 + 0 \cdot V_2 + \cdots + 0 \cdot V_k$, then $\vec{0} \in \text{Span}(V_1, \dots, V_k)$. That is, $\text{Span}(V_1, \dots, V_k) \neq \Phi$.

(ii) Let $x, y \in \text{Span}(V_1, \dots, V_k)$. Then

$$x = \alpha_1 V_1 + \cdots + \alpha_k V_k$$
$$y = \beta_1 V_1 + \cdots + \beta_k V_k$$

So, $x+y = (\alpha_1+\beta_1) V_1 + \cdots + (\alpha_k+\beta_k) V_k$

$= \gamma_1 V_1 + \cdots + \gamma_k V_k$, $\gamma_i = \alpha_i + \beta_i$ $\forall i = 1, \dots, k$

$\Rightarrow x+y \in \text{Span}(V_1, \dots, V_k)$.

(iii) Let $x \in \text{Span}(V_1, \dots, V_k)$, $\alpha \in \mathbb{R}$.

Then $\alpha x = \alpha (c_1 V_1 + c_2 V_2 + \cdots + c_n V_n)$

$= (\alpha c_1) V_1 + (\alpha c_2) V_2 + \cdots + (\alpha c_n) V_n$

$\Rightarrow \alpha x \in \text{Span}(V_1, \dots, V_k)$.

therefore, $\text{Span}(V_1, \dots, V_k)$ is a subspace of V

<u>Df.</u> (Spanning set). Let V be a vector space. A set $v_1, v_2, ---, v_k \in V$ is called a <u>spanning set</u> iff

$$V = \text{Span}(v_1, ---, v_k).$$

<u>Notation</u>. Let $V = \mathbb{R}^n$ and let e_i be an $n\times 1$ column matrix with 1 in the ith component and zero otherwis

that is, $e_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ ith component.

In $\mathbb{R}^3$, $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

<u>Ex.</u> $\{e_1, e_2, ---, e_n\}$ is the standard spanning set for $\mathbb{R}^n$. Since if $x \in \mathbb{R}^n$,

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + --- + x_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$= x_1 e_1 + x_2 e_2 + -- + x_n e_n$$

for ex. $\left\{ e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ is a spanning set for $\mathbb{R}^2$ since if $x \in \mathbb{R}^2$, then

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = x_1 e_1 + x_2 e_2 .$$

$\Rightarrow \{e_1, e_2\}$ spans $\mathbb{R}^2$ or $\{e_1, e_2\}$ is a spanning set for $\mathbb{R}^2$.

Ex. $1, x, ----, x^{n-1}$ is the standard spanning set for P_n. Since if $f(x) \in P_n$, then

$$f(x) = a_0 + a_1 x + ---+ a_{n-1} x^{n-1} \in P_n$$
$$= a_0(1) + a_1(x) + ---+ a_{n-1}(x^{n-1}) .$$

Ex. Is $V_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $V_2 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $V_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ a spanning set for $\mathbb{R}^3$?

Let $V = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$. and Let $V = c_1 V_1 + c_2 V_2 + c_3 V$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$c_1 + c_2 = a$

$2c_1 + 2c_2 = b$

$3c_1 + 2c_2 + c_3 = c$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 2 & 2 & 0 & b \\ 3 & 2 & 1 & c \end{array}\right] \xrightarrow[-3R_1+R_3]{-2R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 0 & 0 & b-2a \\ 0 & -1 & 1 & c-3a \end{array}\right]$$

this system is <u>not</u> always consistent

$\Rightarrow \{V_1, V_2, V_3\}$ is not a spanning set for $\mathbb{R}^3$.

<u>Ex.</u> Is $V_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $V_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $V_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ a spanning set for $\mathbb{R}^2$?

<u>Sol.</u> Let $V = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$. and $V = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3$

$$\alpha_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$\alpha_1 + \alpha_2 + \alpha_3 = x$

$2\alpha_1 + \alpha_3 = y$

This system is ~~[illegible]~~ always consistent (why?)

$\Rightarrow \{V_1, V_2, V_3\}$ is [redacted] a spanning set.

Ex. Is $V_1 = x$, $V_2 = 1$, $V_3 = 2x-1$ a spanning set for P_3 ?

Sol. let $f(x) = ax^2+bx+c \in P_3$, and let

$$f(x) = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3$$

$$\Rightarrow ax^2+bx+c = \alpha_1 \cdot x + \alpha_2 \cdot 1 + \alpha_3 (2x-1).$$

x^2: $a = 0$

x : $\alpha_1 + 2\alpha_3 = b$

x^0: $\alpha_2 - \alpha_3 = c$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & b \\ 0 & 1 & -1 & c \\ 0 & 0 & 0 & a \end{array}\right]$$

this system is not always consistent.

$\Rightarrow \{V_1, V_2, V_3\}$ is not a spanning set.

Ex. Is $V_1 = x$, $V_2 = 1$, $V_3 = 2x-1$ a spanning set for P_2 ?

Sol. let $V = ax+b \in P_2$ and let

$$V = \alpha_1 V_1 + \alpha_2 V_2 + \alpha_3 V_3$$

$$ax+b = \alpha_1 \cdot x + \alpha_2 \cdot 1 + \alpha_3 (2x-1)$$

$$\alpha_1 + 2\alpha_3 = a$$
$$\alpha_2 - \alpha_3 = b$$

$$\left[\begin{array}{ccc|c} \textcircled{1} & 0 & 2 & a \\ 0 & \textcircled{1} & -1 & b \end{array}\right]$$

this system is always consistent

$\Rightarrow \{V_1, V_2, V_3\}$ is a spanning set for P_2.

Linear System Revisited

Theorem. Let A be an $m\times n$ matrix, and let $Ax=b$ be consistent with x_0 a solution. Then y is a solution of $Ax=b$ iff

$$y = x_0 + z, \text{ where } z \in N(A).$$

Proof. $(\Rightarrow)$ Given that $Ax_0 = b$. Assume that y is a solution of $Ax=b$, i.e, $Ay=b$.

We need to show that $y = x_0 + z$, $z \in N(A)$.

Now, $Ax_0 = b$ and $Ay = b$ give

$$A(y - x_0) = Ay - Ax_0 = b - b = 0$$

$\Rightarrow y - x_0 \in N(A)$, i.e, $y - x_0 = z$, $z \in N(A$

$\therefore y = x_0 + z, \quad z \in N(A)$.

($\Leftarrow$) Conversely, Suppose that $y = x_0 + z$, where $z \in N(A)$ and $Ax_0 = b$.

We need to show that $Ay = b$.

Indeed,
$$\begin{aligned} Ay &= A(x_0 + z) \\ &= Ax_0 + Az \\ &= b + 0, \text{ since } z \in N(A) \text{ and } x_0 \text{ is a sol. of } Ax = b \\ &= b \end{aligned}$$

$\blacksquare$

Therefore, $Ay = b$

3.3 linear Independence

Df. Let V be a vector space. A set of vectors $v_1, v_2, ---, v_k \in V$ is called linearly independent iff $c_1 v_1 + c_2 v_2 + --- + c_n v_n = 0 \Rightarrow c_1 = c_2 = --- = c_n = 0$

Otherwise, they are linearly dependent (that is, if there exist scalars $c_1, c_2, --, c_n$ not all zero such that $c_1 v_1 + c_2 v_2 + -- + c_n v_n = 0$)

Ex. Is $\{ \overset{v_1}{\begin{pmatrix}1\\1\end{pmatrix}}, \overset{v_2}{\begin{pmatrix}1\\2\end{pmatrix}} \}$ lin. indep. ?

Let $c_1 v_1 + c_2 v_2 = \begin{pmatrix}0\\0\end{pmatrix}$. So, we have

$$c_1 \begin{pmatrix}1\\1\end{pmatrix} + c_2 \begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{matrix} c_1 + c_2 = 0 \\ c_1 + 2c_2 = 0 \end{matrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 2 & 0 \end{array}\right] \xrightarrow[-R_1+R_2]{} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right]$$

$$\Rightarrow c_1 = c_2 = 0$$

$\therefore \{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix} \}$ is lin. indep. set.

Ex. Is $v_1 = \begin{pmatrix}1\\0\\1\end{pmatrix}, v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$ lin. indep? lin. dep.?

Sol. Let $c_1 v_1 + c_2 v_2 = 0$

$$\Rightarrow c_1\begin{pmatrix}1\\0\\1\end{pmatrix} + c_2\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Rightarrow c_1 = 0 , \quad c_2 = 0$$

$\therefore$ lin. indep.

Ex. Is $V_1 = \begin{pmatrix}1\\2\end{pmatrix}$, $V_2 = \begin{pmatrix}1\\0\end{pmatrix}$, $V_3 = \begin{pmatrix}1\\1\end{pmatrix}$ lin. indep.? lin. dep.?

sol. Let $c_1V_1 + c_2V_2 + c_3V_3 = 0$

$$\Rightarrow c_1\begin{pmatrix}1\\2\end{pmatrix} + c_2\begin{pmatrix}1\\0\end{pmatrix} + c_3\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\Rightarrow \begin{cases} c_1 + c_2 + c_3 = 0 \\ 2c_1 + c_3 = 0 \end{cases}$$

This system is underdetermined homog. system so it has infinite solutions. So the vectors are linearly dependent.

Ex. Is $V_1 = x$, $V_2 = 1$, $V_3 = 2x-1$ lin. indep.

sol. Let $c_1V_1 + c_2V_2 + c_3V_3 = 0$

$$c_1 x + c_2 \cdot 1 + c_3(2x-1) = 0$$

$$\Rightarrow \begin{cases} c_1 + 2c_3 = 0 \\ c_2 - c_3 = 0 \end{cases}$$. This system is underdetermined homogenous system So

it has infinite solutions. So the vectors are linearly dependent.

Thm ① A set of vectors $V_1, V_2, \dots, V_n$ in $\mathbb{R}^n$ are linearly independent iff the matrix $A = [V_1 \ V_2 \ \cdots \ V_n]_{n\times n}$ is nonsingular (i.e, $|A| \neq 0$).

Proof. $V_1, V_2, \dots, V_n$ are linearly independent iff $\alpha_1 V_1 + \alpha_2 V_2 + \cdots + \alpha_n V_n = 0$ has only the zero solution $(\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0)$ iff A is nonsingular. ▨

Ex. Determine whether or not the vectors

$$\left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\2\\4\end{pmatrix} \right\} \text{ lin. dep. ?}$$

Sol. $$|A| = \begin{vmatrix} 1 & 0 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 4 \end{vmatrix} = -1 \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$$

$$\Rightarrow A \text{ is singular} \xRightarrow{\text{Thm}} \left\{ \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\2\\4\end{pmatrix} \right\}$$ are lin. dep.

Rmk. We use thm① for square matrix A.

ex. $\left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix} \right\}$ are lin. indep. in $\mathbb{R}^2$

since $\begin{vmatrix}1&1\\1&2\end{vmatrix} = 2-1 = 1 \neq 0$ (Thm①).

Since $A = \begin{bmatrix}1&1\\1&2\end{bmatrix}$ is nonsingular $\xrightarrow{\text{Thm①}}$ $\left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\2\end{pmatrix} \right\}$ are lin. indep.

Ex. Are $\{P_1(x), P_2(x), P_3(x)\}$ lin. indep.?

where $P_1(x) = 2x^2+x+8$

$P_2(x) = x^2+8x+7$

$P_3(x) = x^2-2x+3$.

Sol. let $c_1 P_1(x) + c_2 P_2(x) + c_3 P_3(x) = 0$

$c_1(2x^2+x+8) + c_2(x^2+8x+7) + c_3(x^2-2x+3) =$

x² terms: $2c_1 + c_2 + c_3 = 0$

x terms: $c_1 + 8c_2 - 2c_3 = 0$

Constant terms: $8c_1 + 7c_2 + 3c_3 = 0$

let $A = \begin{bmatrix}2&1&1\\1&8&-2\\8&7&3\end{bmatrix}$, $|A| = 2(24+14) - 1(3+16) + 1(7-64) = 0$

$\Rightarrow$ A is singular

$\{P_1(x), P_2(x), P_3(x)\}$ are lin. dep.

Thm②. A set of vectors $V_1, V_2, ---, V_k$ in a vector space V are linearly dependent iff one of them is a linear combination of the remaining set of vectors.

proof. ($\Leftarrow$) Say V_1 is a linear combination of $V_2, V_3, ---, V_k$. that is, there exist constants $\alpha_2, \alpha_3, ---, \alpha_k \in \mathbb{R}$ such that

$$V_1 = \alpha_2 V_2 + \alpha_3 V_3 + --- + \alpha_k V_k$$

$$\Rightarrow -1 V_1 + \alpha_2 V_2 + --- + \alpha_k V_k = 0$$

So, not all $(-1, \alpha_2, ---, \alpha_k)$ zero such that $-V_1 + \alpha_2 V_2 + --- + \alpha_k V_k = 0$.

So, $\{V_1, V_2, ---, V_k\}$ are lin. dep.

($\Rightarrow$) Let $V_1, V_2, --, V_k$ are lin. dep.,

So., $c_1 V_1 + c_2 V_2 + --- + c_k V_k = 0$ has a nonzero solution say $(\alpha_1, \alpha_2, ---, \alpha_k)$ and at least one of the α_i's is nonzero, say, $\alpha_1 \neq 0$

$$\Rightarrow V_1 = -\frac{\alpha_2}{\alpha_1}V_2 - \frac{\alpha_3}{\alpha_1}V_3 + \cdots + -\frac{\alpha_k}{\alpha_1}V_k$$

and so V_1 is a linear combination of $V_2, \cdots, V_k$. ∎

Ex. $\left\{ \underset{V_1}{\begin{pmatrix}1\\0\end{pmatrix}}, \underset{V_2}{\begin{pmatrix}0\\1\end{pmatrix}}, \underset{V_3}{\begin{pmatrix}1\\1\end{pmatrix}} \right\}$ are lin. dep. in $\mathbb{R}^2$

Since $V_3 = V_1 + V_2$ (V_3 is a linear combination of V_1 and V_2).

Ex. The vectors $V_1 = \begin{pmatrix}1\\-1\\2\end{pmatrix}$, $V_2 = \begin{pmatrix}-2\\3\\1\end{pmatrix}$, $V_3 = \begin{pmatrix}-1\\3\\8\end{pmatrix}$ are lin. dep. in $\mathbb{R}^3$

Since $V_3 = 3V_1 + 2V_2$ (check!).

Notice that $\text{Span}(V_1, V_2, V_3) = \text{Span}(V_1, V_2)$

$\text{Span}(V_1, V_2, V_3) = \text{Span}(V_1, V_3)$

$\text{Span}(V_1, V_2, V_3) = \text{Span}(V_2, V_3)$

Ex. $\left\{ \underset{V_1}{\begin{pmatrix}1\\1\end{pmatrix}}, \underset{V_2}{\begin{pmatrix}2\\2\end{pmatrix}}, \underset{V_3}{\begin{pmatrix}3\\3\end{pmatrix}} \right\}$ are lin. dep

Since $2V_1 =$ [illegible] V_2 also $3V_1 =$ [illegible] V_3

Notice that if $S = \text{span}\left\{ \overset{v_1}{\begin{pmatrix}1\\1\end{pmatrix}}, \overset{v_2}{\begin{pmatrix}2\\2\end{pmatrix}}, \overset{v_3}{\begin{pmatrix}3\\3\end{pmatrix}} \right\}$,

then $S = \text{span}(v_1)$

$S = \text{span}(v_2)$

$S = \text{span}(v_3)$.

ex. $\{ \overset{P_1}{1}, \overset{P_2}{x}, \overset{P_3}{2-5x} \}$ are lin. dep. since

$$P_3 = 2-5x = 2 \cdot 1 + -5 \cdot x$$
$$= 2P_1(x) - 5P_2(x).$$

Moreover, $\text{Span}\{P_1, P_2, P_3\} = \text{Span}\{1, x\}$

$\text{Span}\{P_1, P_2, P_3\} = \text{Span}\{1, 2-5x\}$

$\text{Span}\{P_1, P_2, P_3\} = \text{Span}\{x, 2-5x\}$.

<u>Thm ③</u>. A set of vectors $v_1, v_2, ---, v_k$ in a vector space V are linearly independent iff every vector $v \in \text{Span}(v_1, ---, v_k)$ is <u>uniquely</u> written as a linear combination of $v_1, v_2, ---, v_k$.

Proof. ($\Rightarrow$) Suppose that $V_1, V_2, ---, V_k$ are lin. indep. (that is $c_1V_1 + --- + c_kV_k = 0$ has the only zero solution $c_1 = c_2 = --- = c_k =$

and suppose that $V \in \text{Span}(V_1, ---, V_k)$ is not unique, say, $V = \alpha_1 V_1 + --- + \alpha_k V_k$

$$= \beta_1 V_1 + --- + \beta_k V_k$$

$$\Rightarrow (\alpha_1 - \beta_1) V_1 + ---- + (\alpha_k - \beta_k) V_k = 0$$

$\Rightarrow \alpha_1 - \beta_1 = 0, ----, \alpha_k - \beta_k = 0$ since $V_1, V_2, ---, V_k$ are lin. indep.

So, $\alpha_i = \beta_i, \ \forall i = 1, ---, k$.

($\Leftarrow$) let $\alpha_1 V_1 + --- + \alpha_k V_k = 0$.

Since $0V_1 + 0V_2 + --- + 0V_k = 0$, it follows $0 \in \text{Span}(V_1, ---, V_k)$. So, 0 is written as a linear combination of $V_1, ---, V_k$ by two ways:

$$0 = \alpha_1 V_1 + --- + \alpha_k V_k$$

$$0 = 0V_1 + --- + 0V_k.$$

By assumption, $\alpha_1 = 0, \alpha_2 = 0, --- \alpha_k = 0$

thus, $\{V_1, ---, V_k\}$ are lin. indep

The vector space $C^{n-1}[a,b]$.

Df: Let $f_1, f_2, ---, f_n \in C^{n-1}[a,b]$, define $W(f_1, f_2, --- f_n)$ on $[a,b]$ by

$$W(f_1, --- f_n) = \begin{vmatrix} f_1 & ---- & f_n \\ f_1' & ---- & f_n' \\ \vdots & & \\ f_1^{(n-1)} & --- & f_n^{(n-1)} \end{vmatrix}, \text{ then}$$

$W(f_1, ---, f_n)$ is called the Wronskian of $f_1, ---, f_n$.

Ex. Find $W(1,x)$.

$$W(1,x) = \begin{vmatrix} 1 & x \\ 0 & 1 \end{vmatrix} = 1$$

Ex. $$W(1,x,x^2) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2.$$

Ex. $$W(x^2, x|x|) = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix}$$

$$= 2x^2|x| - 2x^2|x|$$

$$= 0, \forall x.$$

<u>Thm</u> ④ Let $f_1, ---, f_n \in C^{n-1}[a,b]$. If there exists a point $x_0 \in [a,b]$ such that

$W(f_1, f_2, --, f_n)(x_0) \neq 0$, then

$\{f_1, ---, f_n\}$ are linearly independent

<u>Proof:</u> If $f_1, f_2, ---, f_n$ were linearly dependent, then there exist scalars $c_1, c_2, ---, c_n$ not all zero such that

$$(*) \quad \boxed{c_1 f_1(x) + c_2 f_2(x) + --- + c_n f_n(x) = 0 \ , \forall x \in [a,}$$

Taking the derivative with respect to x of both sides of (*):

$$c_1 f_1'(x) + c_2 f_2'(x) + --- + c_n f_n'(x) = 0$$

If we continue taking derivatives of both sides, we end up with the system

$$(**) \begin{cases} c_1 f_1(x) + c_2 f_2(x) + --- + c_n f_n(x) = 0 \\ c_1 f_1'(x) + c_2 f_2'(x) + --- + c_n f_n'(x) = 0 \\ \vdots \\ c_1 f_1^{(n-1)}(x) + c_2 f_2^{(n-1)}(x) + --- + c_n f_n^{(n-1)}(x) = 0, \end{cases}$$

for each fixed $x \in [a,b]$

the matrix equation

$$(**)\quad \begin{bmatrix} f_1(x) & --- & f_n(x) \\ f_1'(x) & --- & f_n'(x) \\ \vdots & & \vdots \\ f_1^{(n-1)}(x) & --- & f_n^{(n-1)}(x) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

will have the nontrivial solution $(c_1, c_2, --, c_n)^T$, then the coefficient matrix in (**) would be singular for each $x \in [a,b]$ and hence

$$W(f_1,--,f_n)(x) = \begin{vmatrix} f_1(x) & --- & f_n(x) \\ \vdots & & \vdots \\ f_1^{(n-1)}(x) & --- & f_n^{(n-1)}(x) \end{vmatrix} = 0 \quad \forall x \in [a,b]$$

which is a contradiction. ⊟

Rmk: ① If $f_1, --, f_n$ are lin. dep., then $W(f_1, --, f_n)(x) = 0, \ \forall x \in [a,b]$.

② the converse of thm ④ is not true in general. That is, if $W(f_1, --, f_n) = 0$ for all $x \in [a,b]$, we can not say anything about dep. or indep.

Counterexample $W(x^2, x|x|)$ on $[-1,1]$.

$$W(x^2, x|x|) = \begin{vmatrix} x^2 & x|x| \\ 2x & 2|x| \end{vmatrix} = 0, \ \forall x$$

Thm ④ fails. So we use the definition.

let $c_1 x^2 + c_2 x|x| = 0$, $x \in [-1,1]$.

$$\left.\begin{array}{ll} x=1: & c_1 + c_2 = 0 \\ x=-1: & c_1 - c_2 = 0 \end{array}\right\} \Rightarrow c_1 = c_2 = 0$$

$\Rightarrow \{x^2, x|x|\}$ lin. indep.

<u>Ex.</u> $\{1, x, x^2\}$ are lin. indep. on $(-\infty,\infty)$.

since $W(1,x,x^2) = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = (1)(1)(2)$
$= 2 \neq 0, \forall x.$

$W(2020) = 2 \neq 0 \Rightarrow \{1, x, x^2\}$ lin. indep

<u>Ex.</u> $\{e^x, e^{-x}\}$ are lin. indep. on $(-\infty,\infty)$.

since $W(e^x, e^{-x}) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1-1 = -2 \neq 0$

$W(e^x, e^{-x})(4) = -2 \neq 0 \Rightarrow$ lin. indep.

Ex. $\{x, x\ln x\}$ on $(0,\infty)$.

$W(x, x\ln x) = \begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x + x\ln x - x\,l$
$= x$

$W(x, x\ln x)(4) = 4 \neq 0 \Rightarrow \{x, x\ln x\}$ lin. indep

3.4 Basis and Dimension

<u>Df.</u> A set of vectors $V_1, V_2, ---, V_n$ form a <u>basis</u> for a vector space V iff

(i) $V_1, V_2, ---, V_n$ span V.

(ii) $V_1, V_2, ---, V_n$ are linearly independent

<u>Examples</u> ① $\{e_1, e_2, ---, e_n\}$ is a basis for $\mathbb{R}^n$ and it is called the <u>standard basis</u>

for example $\{e_1, e_2, e_3\}$ form a basis for $\mathbb{R}^3$ since

(ii) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0 \Rightarrow \{e_1, e_2, e_3\}$ lin. indep

(i) let $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$. Then

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= ae_1 + be_2 + ce_3$$

$\Rightarrow \{e_1, e_2, e_3\}$ span $\mathbb{R}^3$.

② $\{1, x, x^2, ---, x^{n-1}\}$ is a basis for P_n called the standard basis.

③ $E_{ij} = (e_{ij})$, where $e_{ij} = 1$ and 0 otherwise is a standard basis for $\mathbb{R}^{m\times n}$. For example, the standard basis for $\mathbb{R}^{2\times 3}$ is

$$\left\{ \begin{bmatrix}1&0&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&1&0\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&1\\0&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\1&0&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&1&0\end{bmatrix}, \begin{bmatrix}0&0&0\\0&0&1\end{bmatrix} \right\}$$

④ $V_1 = \begin{pmatrix}1\\2\\3\end{pmatrix}$, $V_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}$, $V_3 = \begin{pmatrix}1\\0\\0\end{pmatrix}$ is a basis for $\mathbb{R}^3$ since

(i) Let $\begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3$ such that

$$\begin{pmatrix}a\\b\\c\end{pmatrix} = \alpha_1\begin{pmatrix}1\\2\\3\end{pmatrix} + \alpha_2\begin{pmatrix}1\\1\\0\end{pmatrix} + \alpha_3\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= a \\ 2\alpha_1 + \alpha_2 &= b \\ 3\alpha_1 &= c \end{aligned}$$

this system is always consistent (why?)

$\Rightarrow \{V_1, V_2, V_3\}$ is spanning set.

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 0 \end{vmatrix} = 1(0) - 1(0) + 1(0-3)$

$= -3 \neq 0$

$\Rightarrow \{V_1, V_2, V_3\}$ are lin. indep.

⑤ $\{1+x, x\}$ is a basis for P_2. Since

(i) Let $f(x) = ax+b = \alpha_1(1+x) + \alpha_2(x)$.

$$\Rightarrow \begin{aligned} \alpha_1 + \alpha_2 &= a \\ \alpha_1 &= b \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 1 & a \\ 1 & 0 & b \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & a \\ 0 & -1 & b-a \end{array}\right]$$

is always consistent

$\therefore \{1+x, x\}$ span P_2.

(ii) Let $\alpha_1(1+x) + \alpha_2(x) = 0$

$\Rightarrow \alpha_1 = 0 \quad , \alpha_1 + \alpha_2 = 0$

$\Rightarrow \alpha_2 = 0$

$\therefore \{1+x, x\}$ lin. indep.

Def'n. Let V be a nonzero vector space. If V has a finite basis $v_1, v_2, \ldots, v_n$, then V is called finite dimensional vector space with dimension n, written $\boxed{\dim V = n}$. The zero vector space $\{\vec{0}\}$ has dimension zero with basis ϕ.

Otherwise, V is called infinite dimensional, written $\boxed{\dim V = \infty}$.

Examples.

1) $\dim \mathbb{R}^n = n$ (finite dim.)

2) $\dim P_n = n$ (finite dim)

3) $\dim \mathbb{R}^{m\times n} = m\cdot n$ (finite dim)

4) $\dim \{\vec{0}\} = 0$ (finite dim).

5) $\dim \mathbb{R} = 1$ (finite dim)

6) $\dim C^n[a,b] = \infty$. (infinite dim.)

Ex. Find a basis and dimension for $N(A)$, where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}_{2\times 3}$.

Sol. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{array}\right]$ (Recall $N(A) = \{x \in \mathbb{R}^3 : Ax = 0\}$

$$\xrightarrow[-2R_1+R_2]{} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|c} \textcircled{1} & 0 & -1 & 0 \\ 0 & \textcircled{1} & 2 & 0 \end{array}\right]$$

x_1, x_2 leadings, $\boxed{x_3 = t}$ free

$$\begin{cases} x_1 - x_3 = 0 \\ x_2 + 2x_3 = 0 \end{cases} \Rightarrow \begin{array}{l} \boxed{x_1 = t} \\ \boxed{x_2 = -2t} \end{array}$$

$$\therefore N(A) = \left\{ \begin{pmatrix} t \\ -2t \\ t \end{pmatrix} : t \in \mathbb{R} \right\} = \left\{ t \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

A basis for $N(A)$ is $\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$

and $\dim N(A) = 1$.

Ex. Find a basis and dimension of

$$S = \left\{ \begin{pmatrix} a-b+c \\ 2b-3c \\ 4a+2c \end{pmatrix} : a,b,c \in \mathbb{R} \right\}$$

Solution.

Let $X \in S \Rightarrow X = \begin{pmatrix} a \\ 0 \\ 4a \end{pmatrix} + \begin{pmatrix} -b \\ 2b \\ 0 \end{pmatrix} + \begin{pmatrix} c \\ -3c \\ 2c \end{pmatrix}$

$$= a\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + b\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} + c\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$$

$\Rightarrow \beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \right\}$ Span S.

- linear Independence

$$\begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 4 & 0 & 2 \end{vmatrix} = 1(4-0) + 1(0+12) + 1(0-8)$$

$$= 4 + 12 - 8 = 8 \neq 0$$

$\Rightarrow \beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \right\}$ lin. indep.

$\Rightarrow \beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \right\}$ is a basis for S.

dim S = 3

Ex. Find a basis and dimension of

$$S = \{ (a+3b+c, \ 2a+6b, \ c)^T : a,b,c \in \mathbb{R} \}$$

Sol. Let $x \in S$, then

$$x = a(1,2,0)^T + b(3,6,0)^T + c(1,0,1)^T$$

$$\therefore \ S = \text{Span}\left\{ \underset{V_1}{\begin{pmatrix}1\\2\\0\end{pmatrix}}, \underset{V_2}{\begin{pmatrix}3\\6\\0\end{pmatrix}}, \underset{V_3}{\begin{pmatrix}1\\0\\1\end{pmatrix}} \right\}$$

notice that $V_2 = 3V_1$, then $\{V_1, V_2, V_3\}$ linearly dep.

$$\therefore \ S = \text{Span}\left\{ \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$$

notice that $\left\{ \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$ lin. indep.
(أتحققها)

$\therefore \left\{ \begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix} \right\}$ is a basis for S

and $\dim S = 2$.

Ex. Find a basis and dimension of

$$S = \{ p(x) \in P_3 : \ p(0) = 0 \ \text{and} \ p'(1) = 0 \}$$

Sol. $p(x) \in P_3 \Rightarrow p(x) = ax^2 + bx + c$

$p(x) \in S$: $p(0) = 0 \Rightarrow \boxed{c = 0}$

$p'(x) = 2ax + b$

$0 = p'(1) = 2a + b \Rightarrow \boxed{b = -2a}$

$\therefore p(x) \in S$ iff $p(x) = ax^2 - 2ax$

$= a(x^2 - 2x)$

$= \text{span}\{x^2 - 2x\}$

$\therefore$ A basis of S is $\{x^2 - 2x\}$ lin. indep. check!

$\dim S = 1$

Ex. Find a basis and dimension of

$S = \{p(x) \in P_3 : p''(x) = 0\}$

Sol. $p(x) \in P_3 \Rightarrow p(x) = ax^2 + bx + c$

$p'(x) = 2ax + b, \quad p''(x) = 2a.$

$p(x) \in S$ iff $p''(x) = 0 \Leftrightarrow 2a = 0$

$a = 0$

$\therefore p(x) = bx + c \cdot 1 = \text{span}\{x, 1\}$

- $\{x, 1\}$ is lin. indep. set. since if $\alpha x + \beta \cdot 1 = 0 \Rightarrow \alpha = 0, \beta = 0$.

$\therefore \{x, 1\}$ is a basis for S and $\dim S = 2$.

<u>thm ①</u>. Let $\{v_1, v_2, \dots, v_n\}$ be a spanning set for V. If $w_1, w_2, \dots, w_k \in V$, $k > n$. Then $w_1, w_2, \dots, w_k$ are linearly dependent.

<u>Ex.</u> $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \end{pmatrix} \right\}$ is a spanning set for $\mathbb{R}^2$.

Then $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \begin{pmatrix} -7 \\ 3 \end{pmatrix} \right\}$ are linearly dep. by thm ①.

<u>thm ②</u>. Let $\{v_1, \dots, v_n\}$ and $\{w_1, \dots, w_k\}$ be two bases for a vector space V. then

$k = n$.

Thm③. Let V be a vector space with $\dim V = n$ then the following are equivalent.

(i) $\{V_1, ---, V_n\}$ is a basis.

(ii) $\{V_1, ---, V_n\}$ Span V

(iii) $\{V_1, ---, V_n\}$ linearly indep.

Ex. $S = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \end{pmatrix} \right\}$ is a basis for $\mathbb{R}^2$

Since $\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} = 5 - 8 \neq 0$ (lin. indep.)

Thm③ $\Rightarrow$ S is a basis for $\mathbb{R}^2$.

Rmk (Summary) Let V be a vector space with $\dim V = n > 0$. then

1) A set $V_1, ---, V_k$, $k > n$ linearly dep.

2) A set $V_1, ---, V_k$, $k < n$ Can not span V.

3) If $k = n$ and $V_1, ---, V_k$ are lin. indep. or span V, then $\{V_1, V_2, --, V_k\}$ is a basis for V.

4) A spanning set of $V_1, V_2, \dots, V_k$, $k > n$ can be reduced (pared down) to a basis for V.

5) A linearly independent set $V_1, \dots, V_k$, $k < n$ can be extended to a basis for V.

Ex. (Q10) $x_1 = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, $x_2 = \begin{pmatrix} 2 \\ 5 \\ 4 \end{pmatrix}$, $x_3 = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$, $x_4 = \begin{pmatrix} 2 \\ 7 \\ 4 \end{pmatrix}$, $x_5 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ span $\mathbb{R}^3$.

pare down $\{x_1, x_2, x_3, x_4, x_5\}$ to form a basis for $\mathbb{R}^3$.

Sol.

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_5 \end{array} \\ \begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 1 \\ 2 & 4 & 0 \end{vmatrix} \end{array} = 1(0-4) - 2(0-2) + 1(8-10)$$

$$= -4 + 4 - 2 = -2 \neq 0$$

$\Rightarrow \{x_1, x_2, x_5\}$ is a basis for $\mathbb{R}^3$.

ex. $\left\{ \overset{f_1}{x}, \overset{f_2}{x-1}, \overset{f_3}{x+1}, \overset{f_4}{x^2-1} \right\}$ Span P_3.

paredown $\{f_1, f_2, f_3, f_4\}$ to form a basis of P_3.

Sol. $W(f_1, f_2, f_4) = \begin{vmatrix} x & x-1 & x^2-1 \\ 1 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix} = 2$

$W(f_1, f_2, f_4)(1) = 2 \neq 0$

$\therefore \{f_1, f_2, f_4\}$ lin. indep.

Rmk $\Longrightarrow$ $\{f_1, f_2, f_4\}$ form a basis for P_3.

Ex. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \right\}$ is lin. indep.

We can extend it to a <u>basis</u> for $\mathbb{R}^3$

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} \right\}$$

Since $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6 \neq 0$ (lin. indep.).

3.5 change of basis

Df. Let V be a vector space and let $E = \{v_1, v_2, ---, v_n\}$ be a basis of V. then any $v \in V$ can be written uniquely as

$$v = \alpha_1 v_1 + \alpha_2 v_2 + --- + \alpha_n v_n,$$ where

$\alpha_1, \alpha_2, ---, \alpha_n$ are scalars. The vector $\boxed{\vec{\alpha} = (\alpha_1, \alpha_2, ---, \alpha_n)^T \in \mathbb{R}^n}$ is called the coordinate of v with respect to a basis E denoted by $[v]_E$ or v_E. that is,

$$[v]_E = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

Ex. Let $v = \begin{pmatrix} 2 \\ 5 \end{pmatrix} \in \mathbb{R}^2$ and let $E = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ be a basis for $\mathbb{R}^2$ (standard basis). Find $[v]_E$.

Sol. Let $v = \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\Rightarrow \alpha_1 = 2, \quad \alpha_2 = 5$$

$$\therefore [v]_E = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} = v.$$

Rmk. If E is a standard basis for a vector space $\mathbb{R}$

then $[v]_E = v$, $\forall\, v \in \mathbb{R}^n$.

Ex. let $f(x) = x^2+2 \in P_3$ and let $E = \{1, x, x^2\}$ be the standard basis for P_3. Find $[f(x)]_E$.

Sol. let $f(x) = \alpha_1 \cdot 1 + \alpha_2 \cdot x + \alpha_3 \cdot x^2$

$$x^2+2 = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$

$$\Rightarrow \boxed{\alpha_3 = 1},\ \boxed{\alpha_1 = 2},\ \boxed{\alpha_2 = 0}$$

$$\therefore [f(x)]_E = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Ex. let $E = \left\{ \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ be a basis for $\mathbb{R}^2$.

Let $x = \begin{bmatrix} 7 \\ 4 \end{bmatrix}$. Find $[x]_E$.

Sol. let $x = \alpha_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} 7 \\ 4 \end{pmatrix} = \alpha_1 \begin{pmatrix} 3 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow (*) \begin{cases} 3\alpha_1 + \alpha_2 = 7 \\ 2\alpha_1 + \alpha_2 = 4 \end{cases} \Rightarrow \boxed{\alpha_1 = 3},\ \boxed{\alpha_2 = -2}$$

$$\therefore [x]_E = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

Another method back to the system (*)

$$\underbrace{\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}}_{U}\underbrace{\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}}_{\alpha}=\underbrace{\begin{bmatrix}7\\ 4\end{bmatrix}}_{X}$$

$$\Rightarrow \begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix}=U^{-1}X=\begin{bmatrix}3 & 1\\ 2 & 1\end{bmatrix}^{-1}\begin{bmatrix}7\\ 4\end{bmatrix}$$

$$=\frac{1}{3-2}\begin{bmatrix}1 & -1\\ -2 & 3\end{bmatrix}\begin{bmatrix}7\\ 4\end{bmatrix}=\begin{bmatrix}3\\ -2\end{bmatrix}$$

$$\therefore [X]_E=\begin{bmatrix}3\\ -2\end{bmatrix}.$$

Df. U in the last example is called the transition matrix from the basis $E=\{u_1,u_2\}$ to the standard basis $\{e_1,e_2\}$.

$$\begin{tikzcd}
{[u_1,u_2]} \arrow[r, "U_1"] \arrow[dr, "S=U_2^{-1}U_1"'] & {[e_1,e_2]} \arrow[d, "U_2^{-1}"] \\
 & {[v_1,v_2]}
\end{tikzcd}$$

Ingeneral, if $[u_1,u_2]$, $[v_1,v_2]$ are any two none standard basis of $\mathbb{R}^2$.

Let $U_1 = (u_1, u_2)$ be the transition matrix from $[u_1, u_2]$ into the standard basis $[e_1, e$

$U_2 = (v_1, v_2)$ be the transition matrix from $[v_1, v_2]$ into the standard basis $[e_1, e_2]$.

Then the transition matrix from $[u_1, u_2]$ into $[v_1, v_2]$ is $S = U_2^{-1} U_1$.

Thm: Let V be a finite dimensional vector space with $\dim V = n$. If $E = [v_1, v_2, \dots, v_n]$, $F = [w_1, w_2, \dots, w_n]$ be two bases. Then the transition matrix from the basis E into the basis F is the $n \times n$ nonsingular matrix

$$S_{E \to F} = \left([v_1]_F, [v_2]_F, \dots, [v_n]_F \right).$$

Rmk: Let V be a vector space with $\dim V < +\infty$ and let E be a basis for V, and $v_1, v_2, \dots, v_k \in V$.

Then (i) $[v_1 + v_2 + \dots + v_k]_E = [v_1]_E + [v_2]_E + \dots + [v_k]_E$

(ii) $v_1, v_2, \dots, v_k$ are lin. indep. iff $[v_1]_E, \dots, [v_k]_E$ are lin. indep.

Ex. Let $E = \left\{ \overset{u_1}{\begin{pmatrix} 5 \\ 2 \end{pmatrix}}, \overset{u_2}{\begin{pmatrix} 7 \\ 3 \end{pmatrix}} \right\}$ and $F = \left\{ \overset{v_1}{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}, \overset{v_2}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \right\}$ be two bases for $\mathbb{R}^2$. Find the transition matrix S from E to F.

Sol. $S_{E\to F} = U_2^{-1} U_1$

$$= \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix}$$

$$= \frac{1}{3-2} \begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 5 & 7 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ -4 & -5 \end{bmatrix}$$

Rmk. $[x]_F = S_{E\to F} [x]_E = (U_2^{-1} U_1) [x]_E$.

Ex. Let $E = [\overset{u_1}{3x+6}, \overset{u_2}{9}]$, $F = [\overset{v_1}{2x+1}, \overset{v_2}{x-4}]$ be two ordered bases for P_2. Find

(a) the transition matrix from E to F.

(b) Use Part(a) to find $[3x+15]_F$.

Solution: (a) $S_{E\to F} = U_2^{-1} U_1$

$$= \begin{bmatrix} 1 & -4 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6 & 9 \\ 3 & 0 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 1 & 4 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 6 & 9 \\ 3 & 0 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 6+12 & 9+0 \\ -12+3 & -18+0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$$

(b) $[3x+15]_F = S_{E\to F} \cdot [3x+15]_E$

$$= \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} [3x+15]_E$$

Now, $[3x+15]_E = ??$

Let $3x+15 = \alpha_1(3x+6) + \alpha_2(9)$

$\Rightarrow \quad 3\alpha_1 = 3 \Rightarrow \alpha_1 = 1$

$6\alpha_1 + 9\alpha_2 = 15 \Rightarrow \boxed{\alpha_2 = 1}$

$\therefore [3x+15]_E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore [3x+15]_F = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+1 \\ -1-2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$

3.6 Row space and Column space

Df. let A be m×n matrix. then

1) the row space of A is the subspace of $\mathbb{R}^{1\times n}$ spanned by the row vectors of A, that is $\boxed{R(A) = \text{span}(\vec{a_1}, \vec{a_2}, ---, \vec{a_m})}$

2) The column space of A is the subspace of $\mathbb{R}^m$ spanned by the column vectors of A denoted by C(A), that is,

$$\boxed{C(A) = \text{Span}(a_1, a_2, ---, a_n)}.$$

3) The null space of A is the subspace of $\mathbb{R}^n$ which is the solution of the homogeneous system $Ax=0$ denoted by N(

That is, $\boxed{N(A) = \{x \in \mathbb{R}^n : Ax = 0\}}$.

4) The nullity of A denoted by Null(A) is $\boxed{\text{Null}(A) = \dim N(A)}$.

5) The rank of A is $\boxed{\text{rank}(A) = \dim C(A)}$

Thm ①. Let A, B be $m \times n$ equivalent matrices. Then $R(A) = R(B)$. (that is, Two row equivalent matrices have the same row space)

Thm ②. If A is an $m \times n$ matrix, then
$$\dim R(A) = \dim C(A).$$

Thm ③. (Rank-Nullity Thm)
If A is an $m \times n$ matrix. then
$$\boxed{\text{Rank}(A) + \text{Null}(A) = n}.$$

Rmk ① To find rank(A), we do
(i) U is the REF or RREF of A
(ii) The nonzero rows in U form a basis for $R(A)$.

② Let A be $m \times n$ matrix. If U is the REF of A, then the columns of A that correspond to the leading 1's in U is a basis for $C(A)$

Examples.

Ex① Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{2\times3}$. Find $R(A)$ and $C(A)$.

Sol. $R(A) = \text{Span}\{(1,0,0),(0,1,0)\}$

$$= \{\alpha(1,0,0) + \beta(0,1,0) : \alpha,\beta \in \mathbb{R}\}$$

$$= \{(\alpha,\beta,0) : \alpha,\beta \in \mathbb{R}\}.$$

$$C(A) = \text{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\0\end{pmatrix}\right\}$$

$$= \left\{\alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} + \gamma\begin{pmatrix}0\\0\end{pmatrix} : \alpha,\beta,\gamma \in \mathbb{R}\right\}$$

$$= \left\{\begin{pmatrix}\alpha\\\beta\end{pmatrix} : \alpha,\beta \in \mathbb{R}\right\} = \mathbb{R}^2.$$

Ex.②. Let $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}_{3\times4}$.

a) Find a basis for $R(A)$, $C(A)$, $N(A)$

b) nullity(A) "Null(A)"

c) rank(A)

d) the dependency relation.

Solution. $A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & 4 & -3 & 0 \\ 1 & 2 & 1 & 5 \end{bmatrix}$

$$\xrightarrow[-R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow[-R_2]{} \begin{bmatrix} 1 & 2 & -1 & 1 \\ 0 & 0 & \enclose{circle}{1} & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$$\xrightarrow[-2R_2+R_3]{R_1+R_2} \begin{bmatrix} \enclose{circle}{1} & 2 & 0 & 3 \\ 0 & 0 & \enclose{circle}{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

a) A basis for $R(A)$ is $\{ \overset{\vec{u}_1}{(1, 2, 0, 3)}, \overset{\vec{u}_2}{(0, 0, 1, 2)} \}$

A basis for $C(A)$ is $\left\{ \underset{a_1}{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}, \underset{a_3}{\begin{pmatrix} -1 \\ -3 \\ 1 \end{pmatrix}} \right\}$

A basis for $N(A) = ??$

$N(A) = \{ x : Ax = 0 \}$

x_1, x_3 leading variables, $\boxed{x_2 = t}$ $\boxed{x_4 = r}$ free variables back to U: $x_1 + 2x_2 + 3x_4 = 0 \Rightarrow x_1 = -2t - 3r$

$x_3 + 2x_4 = 0 \Rightarrow x_3 = -2r$

$$\therefore N(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2t-3r \\ t \\ -2r \\ r \end{pmatrix} \right\}$$

$$= \left\{ t\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$$

$$= \text{Span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$$

- check $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$ lin. indep.

$\therefore$ A basis for $N(A)$ is $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$.

b) nullity$(A) = \dim N(A) = 2$

rank$(A) = \dim R(A) = \dim C(A) = 2$

Notice that rank(A) + nullity$(A) = 2 + 2 = 4 = n$

c) the dependency relation.

notice that In U, we notice that

$u_2 = 2u_1$

$u_4 = 3u_1 + 2u_3$

$\Rightarrow$ In A, $\boxed{a_2 = 2a_1}$ $\boxed{a_4 = 3a_1 + 2a_3}$ } this is called the dependency relation.

Back to the linear system $Ax = b$

Recall, the consistency theorem

$Ax = b$ is consistent iff b is a linear combination of the columns of A iff $b \in C(A)$.

Thm ④. Let A be $m \times n$ matrix, $b \in \mathbb{R}^m$. Then (i) the linear system $Ax = b$ is consistent for every $b \in \mathbb{R}^m$ iff $C(A)$ span $\mathbb{R}^m$.

(ii) The linear system $Ax = b$ has at most one solution, $\forall\, b \in \mathbb{R}^m$ iff $C(A)$ are linearly independent.

__Corollary__. An $n\times n$ matrix A is nonsingular iff $C(A)$ form a basis for $\mathbb{R}^n$.

__Exercise 8__: (text book).
Let A be $m\times n$ matrix, $m>n$ and $N(A)=\{0\}$.
(a) Are $C(A)$ linearly indep.? Do they span $\mathbb{R}^m$?
(b) How many solution with the system $Ax=b$ have if $b\notin C(A)$?, if $b\in C(A)$?

Sol. (a) Since $N(A)=\{0\}$, then

$$Ax = x_1a_1+x_2a_2+\cdots+x_na_n=0$$ has only trivial solution $(x_1=x_2=\cdots=x_n=0)$.

$C(A)=\{a_1, a_2, \cdots, a_n\}$ are linearly indep.

Since $\dim C(A) = \overset{\text{معطى}}{n} < m$, then $C(A)$ can not span $\mathbb{R}^m$.

(b) If $b\notin C(A)$, then $Ax=b$ has no solution.

If $b\in C(A)$, then $Ax=b$ is consistent (then

Since, by part (a), $C(A)$ are linearly independen

then, by Thm ④, $Ax=b$ has at most one

Both

Two conditions give that $Ax=b$ has exactly one solution.

CH4 Linear Transformations

4.1 Definition and Examples

Df. A mapping L from a vector space V into a vector space W is said to be linear transformation iff

(1) $L(V_1+V_2) = L(V_1)+L(V_2), \ \forall V_1, V_2 \in V$.

(2) $L(\alpha V) = \alpha L(V), \ \forall V \in V, \ \forall \alpha \in \mathbb{R}$.

Notation. $L: V \longrightarrow W$.

Rmk. If $V=W$, then $L: V \longrightarrow V$ is said to be linear operator.

Ex1. $L: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$, $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$ is a linear transformation (linear operator).

Since (i) let $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \in \mathbb{R}^2$, then

$$L\left[\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix}\right] = L\begin{pmatrix} a+c \\ b+d \end{pmatrix} = \begin{pmatrix} 3(a+c) \\ 3(b+d) \end{pmatrix}$$

$$= \begin{pmatrix} 3a \\ 3b \end{pmatrix} + \begin{pmatrix} 3c \\ 3d \end{pmatrix}$$

$$= L\begin{pmatrix} a \\ b \end{pmatrix} + L\begin{pmatrix} c \\ d \end{pmatrix}$$

(ii) $\forall \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$, $\forall \alpha \in \mathbb{R}$, we have

$$L\left(\alpha\begin{pmatrix} a \\ b \end{pmatrix}\right) = L\begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix} = \begin{pmatrix} 3(\alpha a) \\ 3(\alpha b) \end{pmatrix}$$

$$= \alpha\begin{pmatrix} 3a \\ 3b \end{pmatrix} = \alpha L\begin{pmatrix} a \\ b \end{pmatrix}.$$

$\therefore$ L is a linear operator.

<u>Ex2.</u> $L: C[a,b] \longrightarrow \mathbb{R}$

$L(f(x)) = \int_a^b f(x)\,dx$. Is L a lin. trans.?

<u>Sol.</u> (i) $L(f(x)+g(x)) = \int_a^b (f(x)+g(x))\,dx$

$$= \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

$$= L(f(x)) + L(g(x)),$$

$\forall f, g \in C[a,b]$

(ii) $\forall f \in C[a,b]$, $\forall \alpha \in \mathbb{R}$, we have

$$L(\alpha f(x)) = \int_a^b \alpha f(x)\,dx = \alpha\int_a^b f(x)\,dx = \alpha L(f(x))$$

$\therefore$ L is a L.T.

Ex 3. $L : C^1[a,b] \longrightarrow C[a,b]$

$$L(f(x)) = f'(x).$$

show that L is Linear transformation.

Pf. (i) $\forall f, g \in C^1[a,b]$, we have

$$L(f(x)+g(x)) = (f(x)+g(x))'$$
$$= f'(x) + g'(x)$$
$$= L(f(x)) + L(g(x)).$$

(ii) $\forall f \in C^1[a,b]$, $\forall \alpha \in \mathbb{R}$, we have

$$L(\alpha f(x)) = (\alpha f(x))' = \alpha f'(x) = \alpha L(f(x)).$$

Theorem(1) $L : V \longrightarrow W$ is a linear transformation iff $\boxed{L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2)}$

$\forall v_1, v_2 \in V$, α, β scalars ; V, W vector spaces

Ex4. $L : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $L\begin{pmatrix} x \\ y \end{pmatrix} = x + y$. show that L is a linear transformation.

Sol. Let $\alpha, \beta \in \mathbb{R}$, $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \in \mathbb{R}^2$. Then

$$L\left(\alpha\begin{pmatrix} a \\ b \end{pmatrix} + \beta\begin{pmatrix} c \\ d \end{pmatrix}\right) = L\begin{pmatrix} \alpha a + \beta c \\ \alpha b + \beta d \end{pmatrix}$$

$$= (\alpha a + \beta c) + (\alpha b + \beta d)$$

$$= \alpha(a+b) + \beta(c+d)$$

$$= \alpha L\begin{pmatrix} a \\ b \end{pmatrix} + \beta L\begin{pmatrix} c \\ d \end{pmatrix}.$$

Ex5. $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$.
$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$. Is L a linear transformation.

Ans. No.

$$L\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = L\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$L\begin{pmatrix} 1 \\ 0 \end{pmatrix} + L\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\Rightarrow L\left[\begin{pmatrix}1\\0\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\right] \neq L\begin{pmatrix}1\\0\end{pmatrix}+L\begin{pmatrix}0\\1\end{pmatrix}.$$

Ex 6. Let $L : P_2 \longrightarrow P_3$

$$L(p(x)) = p(x) + x^2.$$

show that L is not a linear transformation.

Sol. Let $p(x) = x+1$, $q(x) = 1-x$

$$L(p(x)+q(x)) = L(x+1+1-x) = L(2)$$
$$= 2+x^2$$

$$L(p(x)) + L(q(x)) = L(x+1) + L(1-x)$$
$$= x+1+x^2 + 1-x+x^2$$
$$= 2+2x^2$$
$$\neq L(p(x)+q(x)).$$

Ex.7. $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$L\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x^2\\y\end{pmatrix}.$$

show that L is not a linear transformatio

Sol: Take $\alpha = 2 \in \mathbb{R}$, $v = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \in \mathbb{R}^2$

$$L(\alpha v) = L\left(2\begin{pmatrix} 1 \\ 5 \end{pmatrix}\right) = L\begin{pmatrix} 2 \\ 10 \end{pmatrix} = \begin{pmatrix} 4 \\ 10 \end{pmatrix}$$

$$\alpha L(v) = 2L\begin{pmatrix} 1 \\ 5 \end{pmatrix} = 2\begin{pmatrix} 1^2 \\ 5 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \end{pmatrix}$$

$$\Rightarrow L\left(2\begin{pmatrix} 1 \\ 5 \end{pmatrix}\right) \neq 2L\begin{pmatrix} 1 \\ 5 \end{pmatrix}.$$

Thm 2. Let V, W be vector spaces, and let $L : V \longrightarrow W$ be a linear transformation. Then

(a) $L(0_V) = 0_W$

(b) $L(v_1 - v_2) = L(v_1) - L(v_2)$.

(c) $L(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n) = \alpha_1 L(v_1) + \cdots + \alpha_n L(v_n)$, $\forall v_1, v_2, \ldots, v_n \in V$, $\forall \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$

Proof: (a) $L(0_V) = L(0\, 0_V) = 0\, L(0_V) = 0_W$

(b) $L(v_1 - v_2) = L(v_1 + -1 v_2)$

$= L(v_1) + L(-1 v_2) = L(v_1) - 1L(v_2)$

$= L(v_1) - L(v_2)$

(c) Exercise (Use Mathematical induction). ∎

Rmk. Let V, W be vector spaces, and let $L: V \to W$ be a mapping. If $L(0_V) \neq 0_W$, then L is not a linear transformation.

Ex. See ex.5 $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$, $L: \mathbb{R}^2 \to \mathbb{R}^3$.

here $V = \mathbb{R}^2$, $W = \mathbb{R}^3$

$$0_V = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad 0_W = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$L(0_V) = L\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \leftarrow 0_W$$

$\therefore L\begin{pmatrix} 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow L$ is not a lin. trans.

Kernel and Images

Df. let $L: V \to W$ be a linear transformation. Then (a) the kernel of L is

$$\boxed{\ker(L) = \{ v \in V : L(v) = 0_W \}}$$

(b) The image (or the range) of L denoted by $\mathrm{Im}(L)$ or $L(V)$ or R_L is defined by

$$\boxed{L(V) = \{ w \in W : w = L(v) \text{ for some } v \in V \}.}$$

(c) If $L(V) = W$, then L is said to be onto.

(d) If $\ker(L) = \{0_V\}$, then L is said to be one-to one (1-1).

OR If $L(v_1) = L(v_2) \Longrightarrow v_1 = v_2$, then L is (1-1)

Ex. 8 let $L : P_3 \longrightarrow \mathbb{R}^2$ be a linear transformation such that

$$L(p(x)) = \begin{pmatrix} p''(x) - p'(1) \\ p(0) \end{pmatrix}$$

(a) Find ker(L) and its dimension.

(b) Find R_L and its dimension.

(c) Is L one to one? justify

(d) Is L onto?

(e) Let $S = P_1$. Find L(S).

Solution. $p(x) \in P_3 \Rightarrow p(x) = ax^2 + bx + c$

$p'(x) = 2ax + b$, $\boxed{p''(x) = 2a}$

$\boxed{p'(1) = 2a + b}$, $\boxed{p(0) = c}$

$$\therefore L(p(x)) = L(ax^2+bx+c) = \begin{pmatrix} p''(x) - p'(1) \\ p(0) \end{pmatrix}$$

$$= \begin{pmatrix} 2a - 2a - b \\ c \end{pmatrix}$$

$$\therefore \boxed{L(ax^2+bx+c) = \begin{pmatrix} -b \\ c \end{pmatrix}}$$

(a) + (c): (a) Let $p(x) \in \ker(L)$, then

$$L(ax^2+bx+c) = \begin{pmatrix} -b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \boxed{b = 0}$, $\boxed{c = 0}$

$\therefore \ker(L) = \{ ax^2 : a \in \mathbb{R} \}$

$= \text{span}\{x^2\}$

A basis for $\ker(L) = \{x^2\}$, $\dim \ker(L) = 1$

(c) Since $\ker(L) \neq \{0\}$, then L is not 1-1.

(b) + (d):

(b) $R_L = \{ \binom{\alpha}{\beta} \in \mathbb{R}^2 : L(p(x)) = \binom{\alpha}{\beta} = \binom{-b}{c}$ for some $p(x) \in P_3 \}$

$L(p(x)) = \binom{-b}{c} = b\binom{-1}{0} + c\binom{0}{1}$

$= \text{span}\{ \binom{-1}{0}, \binom{0}{1} \}$

. $\{ \binom{-1}{0}, \binom{0}{1} \}$ are lin. indep.

since $\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 \neq 0$.

$\therefore \{ \binom{-1}{0}, \binom{0}{1} \}$ is a basis for R_L

which is a basis for $\mathbb{R}^2$.

$\Rightarrow R_L = \mathbb{R}^2$ $(\dim R_L = 2)$

$\therefore L$ is onto.

(e) $L(S) = L(P_1)$, $P_1 = \{ f(x) = \alpha , \alpha \in \mathbb{R} \}$

$$= L(\alpha) = \begin{pmatrix} 0 \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

A basis for $L(S)$ is $\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$.

Rmk. In Ex 8, we notice that

$$\dim \ker(L) + \dim R_L = 1 + 2 = 3 = \dim P_3 .$$

In general, if $L : V \longrightarrow W$ a lin. trans. and $\dim V < +\infty$, then

$$\boxed{\dim \ker(L) + \dim R_L = \dim V}.$$

Ex 9. let $L : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$

$$L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 \\ x_4 \end{pmatrix}$$ be

a linear transformation

Find the kernel and image of L.

Is L 1-1 ? onto ? Justify.

Sol. $\ker(L) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 : L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

but $L\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \quad x_1 + x_2 + x_3 = 0$$
$$x_4 = 0$$

$$\left[\begin{array}{cccc|c} \textcircled{1} & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & \textcircled{1} & 0 \end{array}\right]$$

x_4, x_1 leading , x_2, x_3 free

let $x_2 = t$, $x_3 = r$

$\therefore \; x_1 = -x_2 - x_3 = -t - r$

$x_4 = 0$

$\therefore \ker(L) = \left\{ \begin{pmatrix} -t-r \\ t \\ r \\ 0 \end{pmatrix} : t, r \in \mathbb{R} \right\}$

$= \left\{ t\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + r\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} : t, r \in \mathbb{R} \right\}$

$\ker(L) = \operatorname{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

- $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ are lin. indep. (check!).

$\therefore \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for $\ker(L)$ and $\dim \ker(L) = 2 \neq \{0$

$\Longrightarrow L$ is not 1-1.

$\operatorname{Imm}(L) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : \begin{pmatrix} x \\ y \end{pmatrix} = L \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1+x_2+x_3 \\ x_4 \end{pmatrix} \right.$

$\therefore L \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 \\ x_4 \end{pmatrix}$

$= x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$= \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

- $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ lin. indep

$\therefore$ A basis for $\mathrm{Imm}(L)$ is $\left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$ which is a basis for $\mathbb{R}^2$.

$$\therefore \mathrm{Imm}(L) = \mathbb{R}^2$$

$$\Rightarrow L \text{ is onto.}$$

Ex. 10. If $T: P_2 \to P_2$ is a linear operator with $T(x+1) = 2$ and $T(x-2) = -1$ find $T(-3)$.

Sol. let $\boxed{-3 = \alpha(x+1) + \beta(x-2)}$

$$\Rightarrow \begin{matrix} \alpha + \beta = 0 \\ \alpha - 2\beta = -3 \end{matrix} \Rightarrow \alpha = -1, \ \beta = 1$$

$$\therefore \boxed{-3 = -1(x+1) + 1(x-2)}$$

$$T(-3) = -1\,T(x+1) + 1\,T(x-2)$$

$$= -1(2) + 1(-1)$$

$$= -3$$

Theorem (3). Let $L : V \to W$ be a linear transformation. Then

(i) $\ker(L)$ is a subspace of V.

(ii) R_L is a subspace of W.

proof. (i) (1) $0 \in \ker(L)$ since $L(0_V) = 0_W$

$\therefore \ker(L) \neq \phi$.

(2) let $V_1, V_2 \in \ker(L)$ (that is $L(V_1) = 0$, $L(V_2) = 0$). Then

$$L(V_1 + V_2) = L(V_1) + L(V_2) = 0 + 0 = 0$$

$$\Rightarrow V_1 + V_2 \in \ker(L).$$

(3) let $V \in \ker(L)$ $\alpha \in \mathbb{R}$. Then

$$L(\alpha V) = \alpha L(V)$$

$$= \alpha . 0 \quad \text{since } V \in \ker(L)$$

$$= 0$$

$$\Rightarrow \alpha V \in \ker(L).$$

$\therefore \ker(L)$ is a subspace of V.

(ii) R_L is a subspace of W.

(1) $0 \in R_L$ since $L(0_V) = 0_W$.

$\therefore R_L \neq \Phi$.

(2) Let $w_1, w_2 \in R_L$. Then there exist $v_1, v_2 \in V$ such that

$w_1 = L(v_1)$, $w_2 = L(v_2)$

$\Rightarrow w_1 + w_2 = L(v_1) + L(v_2)$
$= L(v_1 + v_2)$ since L is lin. trans.

$\therefore w_1 + w_2 = L(v_1 + v_2)$, where $v_1 + v_2 \in V$

this means $w_1 + w_2 \in R_L$.

(3) let $\alpha \in \mathbb{R}$, $w \in W$. So, there exists $v \in V$ such that $w = L(v)$. then

$\alpha w = \alpha L(v)$, $v \in V$
$= L(\alpha v)$ since L is lin. trans

$\therefore \alpha w = L(\alpha v)$, $\alpha v \in V$ since V is a vector space

this means $\alpha w \in R_L$ ▤

Exercises. (textbook) (172)

Q4) $L : \mathbb{R}^2 \to \mathbb{R}^2$ lin. opr.

If $L\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}-2\\3\end{pmatrix}$, $L\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}5\\2\end{pmatrix}$,

find $L\begin{pmatrix}7\\5\end{pmatrix}$.

Solution. Let $\boxed{\begin{pmatrix}7\\5\end{pmatrix} = \alpha\begin{pmatrix}1\\2\end{pmatrix} + \beta\begin{pmatrix}1\\-1\end{pmatrix}}$

$$\begin{array}{l}\alpha + \beta = 7\\ 2\alpha - \beta = 5\end{array} \Rightarrow \boxed{\alpha = 4}, \boxed{\beta = 3}$$

$$\therefore \begin{pmatrix}7\\5\end{pmatrix} = 4\begin{pmatrix}1\\2\end{pmatrix} + 3\begin{pmatrix}1\\-1\end{pmatrix}$$

$$\Rightarrow L\begin{pmatrix}7\\5\end{pmatrix} = 4L\begin{pmatrix}1\\2\end{pmatrix} + 3L\begin{pmatrix}1\\-1\end{pmatrix}$$

$$= 4\begin{pmatrix}-2\\3\end{pmatrix} + 3\begin{pmatrix}5\\2\end{pmatrix}$$

$$= \begin{pmatrix}-8\\12\end{pmatrix} + \begin{pmatrix}15\\6\end{pmatrix} = \begin{pmatrix}7\\18\end{pmatrix}$$

Q14) $L : \mathbb{R}^1 \to \mathbb{R}^1$ linear operator

let $L(1) = a$. show that

$$L(x) = ax, \ \forall x \in \mathbb{R}$$

Proof: $L(x) = L(x \cdot 1) = x\,L(1) = xa = ax \ \forall x \in \mathbb{R}$ ∎

Q22) $L: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ x+y \\ x+y+z \end{pmatrix}$$

Is L 1-1 ? onto ?

Sol. $\ker(L) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ x+y \\ x+y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right.$

$\Rightarrow x = y = z = 0$.

$\therefore \ker(L) = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\}$, $\dim \ker(L) = 0$

since $\ker(L) = \{ 0_{\mathbb{R}^3} \}$, then L is 1-1.

$$\mathrm{Imm}(L) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 : \begin{pmatrix} a \\ b \\ c \end{pmatrix} = L\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ x+y \\ x+y+z \end{pmatrix} \right\}$$

$$= \left\{ x\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + z\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$= \mathrm{Span}\left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{lin. indep check!}} \right\}$$

A basis for $\mathrm{Imm}(L)$ is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\dim \mathrm{Imm}(L) = 3 = \dim \mathbb{R}^3 \Rightarrow \mathrm{Imm}(L) = \mathbb{R}^3$

$\therefore L$ is onto.

4.2 Matrix Representation of Linear Transformation.

Thm. If $E=\{v_1, v_2, \ldots, v_n\}$ and $F=\{w_1, w_2, \ldots, w_m\}$ are ordered bases for vector spaces V and W, respectively, then there exists an $m\times n$ matrix A called the matrix representation of L relative to the ordered bases E and F, such that for any $v\in V$,

$$\boxed{[L(v)]_F = A[v]_E}$$

Moreover,

$$A=\left([L(v_1)]_F, [L(v_2)]_F, \ldots, [L(v_n)]_F\right).$$

Example. Find the matrix representation of the following linear transformation.

$\boxed{1}$ $L: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$L\begin{pmatrix} x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix} x_1-x_2-x_3\\ x_1+x_2+x_3\end{pmatrix},$$ with respect to the ordered bases $E=\{e_1, e_2, e_3\}$, $F=\left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}\right\}$

Solution.

$$A = \left([L(e_1)]_F \, , \, [L(e_2)]_F \, , \, [L(e_3)]_F \right)$$

- $L(e_1) = L\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix} = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}-1\\1\end{pmatrix}$

$$\Rightarrow \begin{array}{l} c_1 - c_2 = 1 \\ c_1 + c_2 = 1 \end{array} \Rightarrow \begin{array}{l} c_1 = 1 \\ c_2 = 0 \end{array}$$

$$\therefore L(e_1) = 1\begin{pmatrix}1\\1\end{pmatrix} + 0\begin{pmatrix}-1\\1\end{pmatrix}$$

$$\boxed{[L(e_1)]_F = \begin{bmatrix}1\\0\end{bmatrix}}$$

- $L(e_2) = L\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} = \alpha_1\begin{pmatrix}1\\1\end{pmatrix} + \alpha_2\begin{pmatrix}-1\\1\end{pmatrix}$

$$\left.\begin{array}{l} \alpha_1 - \alpha_2 = -1 \\ \alpha_1 + \alpha_2 = 1 \end{array}\right\} \begin{array}{l} \alpha_1 = 0 \\ \alpha_2 = 1 \end{array}$$

$$\therefore \boxed{[L(e_2)]_F = \begin{bmatrix}0\\1\end{bmatrix}}$$

- $L(e_3) = L\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix} = 0\cdot\begin{pmatrix}1\\1\end{pmatrix} + 1\begin{pmatrix}-1\\1\end{pmatrix}$

$$\therefore \boxed{[L(e_3)]_F = \begin{bmatrix}0\\1\end{bmatrix}}$$

$$\therefore A = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\end{bmatrix}$$

$\boxed{2}$ $L : P_3 \longrightarrow P_2$

$L(f(x)) = f'(x)$ with respect to the bases $E = [x^2, x, 1]$ and $F = [x, 1]$.

Solution. $A = \left([L(x^2)]_F , (L(x))_F , [L(1)]_F \right)$

- $L(x^2) = 2x = \alpha_1 \cdot x + \alpha_2 \cdot 1$

$$\Rightarrow \alpha_1 = 2, \ \alpha_2 = 0$$

$\therefore [L(x^2)]_F = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

- $L(x) = 1 = 0 \cdot x + 1 \cdot 1$

$[L(x)]_F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

- $L(1) = 0 = 0 \cdot x + 0 \cdot 1$

$[L(1)]_F = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$\boxed{3}$ $L : P_2 \longrightarrow \mathbb{R}^2$

$L(p(x)) = \begin{pmatrix} \int_0^1 p(x)\,dx \\ p(0) \end{pmatrix}$ with respect to the standard bases. That is,

$E = [1, x]$, $F = [e_1, e_2] = [\binom{1}{0}, \binom{0}{1}]$

Solution:

$A = \left([L(1)]_F , [L(x)]_F \right)$

- $L(1) = \begin{pmatrix} \int_0^1 1\,dx \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$c_1 = 1, \; c_2 = 1$

$\therefore [L(1)]_F = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

- $L(x) = \begin{pmatrix} \int_0^1 x\,dx \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\therefore [L(x)]_F = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & 0 \end{bmatrix}.$

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$\boxed{4}$ $L : P_3 \longrightarrow P_2$

$L(f(x)) = f'(x) + f(0)$.

with respect to the ordered bases

$E = [x^2, x, 1]$, $F = [2, 1-x]$.

Solution. $A = \left([L(x^2)]_F , [L(x)]_F , [L(1)]_F \right)$

- $L(x^2) = 2x + 0 = 2x = c_1(2) + c_2(1-x)$

$$\left.\begin{array}{r} 2c_1 + c_2 = 0 \\ -c_2 = 2 \end{array}\right\} \Rightarrow \begin{array}{l} c_1 = 1 \\ c_2 = -2 \end{array}$$

$\therefore [L(x^2)]_F = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

- $L(x) = 1 + 0 = 1 = \alpha_1 \cdot 2 + \alpha_2(1-x)$

$$\left.\begin{array}{r} 2\alpha_1 + \alpha_2 = 1 \\ -\alpha_2 = 0 \end{array}\right\} \begin{array}{l} \alpha_1 = \frac{1}{2} \\ \alpha_2 = 0 \end{array}$$

$\therefore [L(x)]_F = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$

- $L(1) = 0 + 1 = 1 = \frac{1}{2} \cdot 2 + 0(1-x)$

$[L(1)]_F = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$

$\therefore A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ -2 & 0 & 0 \end{bmatrix}$.

Ch6. Eigenvalues

6.1 Eigenvalues and Eigenvectors

Df. let A be an $n\times n$ matrix. A scalar λ is said to be an eigenvalue or a characteristic value of A if there exists a nonzero vector v such that $\boxed{Av = \lambda v}$.

The vector v is said to be an eigenvector or characteristic vector belonging to λ.

Ex. let $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\lambda = 3$, then

$$Av = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8-2 \\ 2+1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \lambda v$$

Thus, $\lambda = 3$ is an eigenvalue of A and $v = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigen vector belonging to $\lambda = 3$.

Question. How to find the eigenvalues and corresponding eigenvectors of a square matrix A ?

Ans. The equation $Av = \lambda v$ can be rewritten in the form $\boxed{(A - \lambda I)v = 0,\ v \neq 0}$

thus, λ is an eigenvalue of A iff $(A-\lambda I)v=0$ has a nontrivial solution iff $N(A-\lambda I) \neq \{0\}$ and any nonzero vector in $N(A-\lambda I)$ is an eigenvector belonging to λ.

thus, for an $n\times n$ matrix A, the following are equivalent:

(1) A nonzero vector $v \in \mathbb{R}^n$ is an eigenvector of A corresponding to the eigenvalue $\lambda \in \mathbb{R}$.

(2) $Av = \lambda v$

(3) $(A-\lambda I_n)v = 0$

(4) The homogeneous system $(A-\lambda I_n)v=0$ has a nonzero solution v.

(5) $N(A-\lambda I_n) \neq \{\vec{0}\}$

(6) $(A-\lambda I_n)$ is singular

(7) $\det(A-\lambda I_n)=0$

Rmk: (i) the equation $\det(A-\lambda I_n)=0$ is called the characterstic equation, for the matrix A.

(ii) $P_A(\lambda) = \det(A-\lambda I_n)$ is called the characterstic polynomial of A.

Example. Find the eigenvalues and the corresponding eigenvector of the given matrices

① $A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$.

Solution. the characterstic equation is

$$|A-\lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 2 \\ 3 & -2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)(-2-\lambda) - 6 = 0$$

$$\Rightarrow \lambda^2 - \lambda - 12 = 0$$

$$\Rightarrow (\lambda-4)(\lambda+3) = 0$$

$$\Rightarrow \lambda_1 = 4, \ \lambda_2 = -3$$

Thus, the eigenvalues of A are

$$\boxed{\lambda_1 = 4} \quad \text{and} \quad \boxed{\lambda_2 = -3}$$

- For $\boxed{\lambda_1 = 4}$, to find the eigenvectors belonging to $\lambda_1 = 4$, we must determine $N(A-4I)$.

$$\left[A-4I \,\middle|\, \begin{matrix}0\\0\end{matrix}\right] = \left[\begin{matrix}-1 & 2\\ 3 & -6\end{matrix} \,\middle|\, \begin{matrix}0\\0\end{matrix}\right] \longrightarrow \left[\begin{matrix}+1 & -2\\ 0 & 0\end{matrix} \,\middle|\, \begin{matrix}0\\0\end{matrix}\right]$$

$x_1 = 2x_2$

Let $x_2 = t \Rightarrow x_1 = 2t$

$$N(A-4I) = \left\{ \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}2t\\t\end{pmatrix} \right\} = \operatorname{span}\left\{ \begin{pmatrix}2\\1\end{pmatrix} \right\}$$

hence $\left\{ V_1 = \begin{pmatrix}2\\1\end{pmatrix} \right\}$ is a basis for the eigenspace corresponding to $\lambda_1 = 4$.

- For $\boxed{\lambda_2 = -3}$, we must find $N(A+3I)$:

$$\left[\begin{matrix}6 & 2\\ 3 & 1\end{matrix} \,\middle|\, \begin{matrix}0\\0\end{matrix}\right] \longrightarrow \left[\begin{matrix}3 & 1\\ 0 & 0\end{matrix} \,\middle|\, \begin{matrix}0\\0\end{matrix}\right]$$

$$\longrightarrow \left[\begin{matrix}1 & \frac{1}{3}\\ 0 & 0\end{matrix} \,\middle|\, \begin{matrix}0\\0\end{matrix}\right]$$

$x_1 = -\frac{1}{3}x_2$

$$\therefore\ N(A+3I) = \left\{ \begin{pmatrix}-\frac{1}{3}x_2\\ x_2\end{pmatrix} : x_2 \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left\{ \begin{pmatrix}-\frac{1}{3}\\ 1\end{pmatrix} \right\}$$

$\Rightarrow V_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ is an eigen vector belonging to $\lambda_2 = -3$.

Ex ② $A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 2 \end{bmatrix}$

<u>Solution.</u> The characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -3 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & -3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\begin{vmatrix} -2-\lambda & 1 \\ -3 & 1-\lambda \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 1 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & -2-\lambda \\ 1 & -3 \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left[(-2-\lambda)(1-\lambda)+3\right] + 3\left[2-\lambda-1\right] + (-3+2+\lambda) = 0$$

(أشغلها)

$$\Rightarrow -\lambda(\lambda-1)^2 = 0$$

$\Rightarrow \boxed{\lambda_1 = 0}$, $\boxed{\lambda_2 = \lambda_3 = 1}$ are the eigenvalues of A.

- For $\boxed{\lambda_1 = 0}$, we must find $N(A - 0I) = N(A$

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -3 & 2 & 0 \end{array}\right] \xrightarrow{-R_2+R_1} \left[\begin{array}{ccc|c} \textcircled{1} & -1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & -3 & 2 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{-R_1+R_2 \\ -R_1+R_3}]{} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$$\xrightarrow[-R_2]{} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \end{array}\right]$$

$$\xrightarrow[2R_2+R_3]{R_2+R_1} \left[\begin{array}{ccc|c} \textcircled{1} & 0 & -1 & 0 \\ 0 & \textcircled{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_1, x_2 leading variables

$x_3 = t$ free

$x_1 = x_3 = t$, $x_2 = x_3 = t$.

$\therefore N(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

$\therefore v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\lambda_1 = 0$

Notice that $\dim N(A - 0I) = 1$

- For $\lambda_2 = \lambda_3 = 1$, we must find $N(A-I)$. (eigenspace corresponding to $\lambda_2 = \lambda_3 = 1$)

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \\ 1 & -3 & 1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} \textcircled{1} & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 3x_2 - x_3$

let $x_2 = r$, $x_3 = t$

$x_1 = 3r - t$

$$\therefore N(A-1I) = \left\{ \begin{pmatrix} 3r-t \\ r \\ t \end{pmatrix} : r, t \in \mathbb{R} \right\}$$

$$= \left\{ r\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$= \text{Span}\Big\{ \underbrace{\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}_{\text{lin. indep (check!)}} \Big\}$$

$\therefore$ $v_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ are an eigen-vectors belonging to $\lambda_2 = \lambda_3 = 1$

notice that $\dim N(A-1I)$ is 2.

Ex ③. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$

Sol. The characterstic polynomial is

$$p(\lambda) = |A - \lambda I|$$

$$= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & 4-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix}$$

$$= (2-\lambda)(4-\lambda)(2-\lambda) \quad (\text{why?})$$

- $p(\lambda) = 0 \Rightarrow \boxed{\lambda_1 = \lambda_2 = 2}, \boxed{\lambda_3 = 4}$

are the eigenvalues of A.

- For $\lambda_1 = \lambda_2 = 2$, we must find $N(A-2I)$.

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$$

$2x_2 = 0 \Rightarrow x_2 = 0$.
$x_3 = t$ free
$x_1 = 0$

$$\therefore N(A-2I) = \left\{ \begin{pmatrix} 0 \\ 0 \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$$

$$= \left\{ t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}$$

$$= \operatorname{span}\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\} = \operatorname{span}\{e_3\}$$

$\therefore$ $v_1 = \begin{pmatrix}0\\0\\1\end{pmatrix}$ is an eigenvector correspondin

to $\lambda_2 = \lambda_1 = 2$.

Notice that $\dim N(A-2I) = 1$.

. For $\lambda_3 = 4$, we need to find $N(A-4I)$.

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -2 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} \textcircled{1} & 0 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 0$, $x_3 = 0$, $x_2 = t$ free

$$\therefore N(A-4I) = \left\{ \begin{pmatrix}0\\t\\0\end{pmatrix} : t \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ \begin{pmatrix}0\\1\\0\end{pmatrix} \right\} = \text{span}\{e_2\}$$

An eigenvector corr. to $\lambda_3 = 4$ is $v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$

Notice $\dim N(A-4I) = 1$.

Complex eigenvalues

Recall, the conjugate of the complex number $z = \alpha + \beta i$ is $\bar{z} = \alpha - \beta i$.

Thm. Let A be a square $n \times n$ matrix with real entries, and let λ be an eigenvalue of A with an eigenvector v. then $\bar{\lambda}$ is an eigenvalue of A with eigenvector $\bar{v}$.

Pf. $$A\bar{v} = \bar{A}\bar{v} \quad (\text{since } A \text{ with real entries})$$
$$= \overline{Av}$$
$$= \overline{\lambda v} = \bar{\lambda}\bar{v}$$

$\therefore A\bar{v} = \bar{\lambda}\bar{v}$, that is, $\bar{\lambda}$ is an eigenvalue of A and $\bar{v}$ is the corresponding eigenvector of $\bar{\lambda}$. $\boxtimes$

Ex. ④ Let $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$. Compute the eigenvalues of A and find bases for the corresponding eigenspaces.

Sol. the characterstic eq. is

$$|A-\lambda I|=0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix}=0 \Rightarrow (1-\lambda)^2+4=0$$

$$\Rightarrow 1-\lambda = \pm 2i$$

$$\boxed{\lambda_1 = 1+2i}, \quad \boxed{\lambda_2 = 1-2i}$$

are the eigenvalues of A.

For $\lambda_1 = 1+2i$, we must find $N(A-\lambda_1 I)$

$$\left[\begin{array}{cc|c} 1-1-2i & 2 & 0 \\ -2 & 1-1-2i & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} -2i & 2 & 0 \\ -2 & -2i & 0 \end{array}\right] \xrightarrow{\frac{1}{2}iR_1} \left[\begin{array}{cc|c} 1 & i & 0 \\ -2 & -2i & 0 \end{array}\right]$$

$$\xrightarrow[2R_1+R_2]{} \left[\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = -ix_2$, let $x_2 = r$ free

$x_1 = -ir$

$$\therefore N(A-(1+2i)I) = \left\{ \begin{pmatrix} -ir \\ r \end{pmatrix} : r \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\}$$

$\Rightarrow$ $V_1 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ is an eigen vector corr. to $\lambda_1 = 1+2i$

by last Thm, $V_2 = \overline{V_1} = \begin{pmatrix} i \\ 1 \end{pmatrix}$ is an eigen-vector corresponding to $\lambda_2 = 1-2i = \overline{\lambda_1}$.

The product and Sum of the Eigenvalues

Df. Let A be nxn matrix. then the trace of A denoted by tr(A) is the sum of all entries on the main diagonal.

Ex. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & -5 & 6 \\ 0 & 2 & 8 \end{bmatrix}$, $tr(A) = 1+-5+8$
$= 4$.

Thm. Let A be a square nxn matrix with eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_n$. then

① $\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$.

② $tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.

Ex. verify the last thm for $A = \begin{bmatrix} 5 & -18 \\ 1 & -1 \end{bmatrix}$

(191)

Sol: $P(\lambda) = |A - \lambda I|$

$$= \begin{vmatrix} 5-\lambda & -18 \\ 1 & -1-\lambda \end{vmatrix}$$

$$= (5-\lambda)(-1-\lambda) + 18$$

$$= -5 - 5\lambda + \lambda + \lambda^2 + 18$$

$$\Rightarrow \boxed{P(\lambda) = \lambda^2 - 4\lambda + 13}$$

- $P(\lambda) = 0 \Rightarrow \lambda^2 - 4\lambda + 13 = 0$

$$\lambda = \frac{4 \pm \sqrt{16 - 4(13)}}{2}$$

$$\lambda = 2 \pm 3i$$

$\therefore \lambda_1 = 2+3i$, $\lambda_2 = 2-3i$ are the eigenvalues of A.

- $tr(A) = 5 + -1 = 4$.

$\lambda_1 + \lambda_2 = 2+3i+2-3i = 4 = tr(A)$.

- $det(A) = -5 + 18 = 13$

$\lambda_1 \lambda_2 = (2+3i)(2-3i) = 4+9 = 13 = |A|$.

Thm. let A be an $n\times n$ matrix. then A is singular iff 0 is an eigenvalue of A.

Thm. let A be an $n\times n$ matrix. Then A and A^T have the same eigenvalues.

Thm. let A be an $n\times n$ matrix. If λ is an eigenvalue of A. If $n\in\mathbb{Z}^+$, then λ^n is an eigenvalue of A^n with the same eigenvector.

Ex. Let $A=\begin{pmatrix} 2 & 0 \\ 5 & 2 \end{pmatrix}$

(a) Find the eigenvalue(s) of A and the corr. eigenvectors.

(b) Compute $A^{100}\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

Sol. $\det(A-\lambda I)=0 \Rightarrow \begin{vmatrix} 2-\lambda & 0 \\ 5 & 2-\lambda \end{vmatrix}=0$

$$\Rightarrow (2-\lambda)^2=0$$

$$\Rightarrow \lambda_1=\lambda_2=2$$

- $N(A-2I) = ??$

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 5 & 0 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$x_1 = 0$, $x_2 = r$ free.

$$\Rightarrow N(A-2I) = \left\{ \begin{pmatrix} 0 \\ r \end{pmatrix} \right\}$$
$$= \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$\therefore V_1 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$ is an eigenvector corr. to $\lambda_1 = 2$.

So, we have $AV_1 = \lambda_1 V_1$

$$A\begin{pmatrix} 0 \\ 3 \end{pmatrix} = 2V_1$$

by last thm, 2^{100} is an eigenvalue of A^{100} with the same eigenvector $V_1 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

$$\therefore A^{100}\begin{pmatrix} 0 \\ 3 \end{pmatrix} = 2^{100}\begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \cdot 2^{100} \end{pmatrix}.$$

Thm. Let A be an $n\times n$ nonsingular matrix. If λ is an eigenvalue of A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} with the same eigenvectors.

Similar Matrices.

Df. A matrix B is said to be similar to a matrix A if there exists a nonsingular matrix S such that $\boxed{B = S^{-1}AS}$.

Thm. Let A, B be $n\times n$ matrices. If B is similar to A, then

(a) $|A| = |B|$

(b) A and B have the same characteristic polys. and the same eigenvalues, but need not have the same eigenvectors.

Proof. (a)

$$\begin{aligned} |B| &= |S^{-1}AS| \\ &= |S^{-1}|\,|A|\,|S| \\ &= \frac{1}{|S|}\,|A|\,|S| = |A|. \end{aligned}$$

(195)

(b) $P_B(\lambda) = \det(B - \lambda I)$

$= \det(S^{-1}AS - \lambda I)$

$= \det(S^{-1}(A - \lambda I)S)$

$= |S^{-1}|\,|A - \lambda I|\,|S|$

$= \frac{1}{\cancel{|S|}}\,|A - \lambda I|\,\cancel{|S|}$

$= |A - \lambda I| = P_A(\lambda)$.

$\therefore P_B(\lambda) = 0 \iff P_A(\lambda) = 0$.

hence A and B have the same characteristi
Polynomials and consequently the same
eigenvalues. ▤

6.3 Diagonalization

Thm. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of an $n\times n$ matrix A with corresponding eigenvectors $V_1, V_2, \ldots, V_k$, then $V_1, V_2, \ldots, V_k$ are linearly independent.

Df. An $n\times n$ matrix A is said to be diagonalizable if there exists a nonsingular matrix X and a diagonal matrix D such that $\boxed{X^{-1}AX = D}$ or $\boxed{A = XDX^{-1}}$.

We say that X diagonalizes A.

A matrix that is not diagonalizable is called defective.

Thm. An $n\times n$ matrix A is diagonalizable iff A has n linearly independent eigenvectors.

Rmk. (1) If A is diagonalizable, then the column vectors of the matrix X are eigenvectors of A and the diagonal elements of D are the corresponding eigenvalues of A.

(2) the diagonalizing matrix X is not unique.

(3) If A is $n \times n$ and A has n distinct eigenvalues, then A is diagonalizable.

If the eigenvalues are not distinct, then A may or may not be diagonalizable depending on whether A has n linearly indep. eigenvectors.

(4) If A is diagonalizable, then A can be factored as $A = XDX^{-1}$.

and $A^k = XD^kX^{-1} = X \begin{bmatrix} \lambda_1^k & \cdots & 0 \\ 0 & \lambda_2^k \cdots & 0 \\ \vdots & & \\ 0 & \cdots & \lambda_n^k \end{bmatrix} X^{-1}$

Examples (Back to Ex1 section 6.1 page 181)

$A = \begin{bmatrix} 3 & 2 \\ 3 & -2 \end{bmatrix}$. Is A diagonalizable? If so, find X and D such that $A = XDX^{-1}$!

Sol. $\lambda_1 = 4$, $\lambda_2 = -3$ are the eigenvalues.

$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ are the corresponding eigenvectors to λ_1, λ_2 respectively.

Since A has distinct eigenvalues, then A is diagonalizable.

$$X = \begin{array}{c} \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix} \\ \;\; v_1 \quad v_2 \end{array}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}$$

the eigenvalues

Ex② Section 6.1 page 183.

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & -2 & 1 \\ 1 & -3 & 1 \end{bmatrix}$$

$\lambda_1 = 0$, $\lambda_2 = \lambda_3 = 1$

$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $v_2 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

A is diagonlizable since A has three linearly indep. eigenvectors.

$$X = \begin{bmatrix} 1 & 3 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Ex3 Section 6.1 page 186.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$\lambda_1 = \lambda_2 = 2, \quad \lambda_3 = 4.$

$V_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad V_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

Since A has fewer than 3 linearly indep. eigenvectors, then A is NOT diagonalizable (defective).

(200)

Selected Exercises (Sections 6.1 and 6.3)

Section 6.1 3, 4, 8, 14, 16

Q3) Let A be $n \times n$ matrix. Prove that A is singular iff $\lambda = 0$ is an eigenvalue of A.

proof: A is singular iff $\det(A) = 0$.

$\lambda = 0$ is an eigenvalue of A iff

$$\det(A - 0I) = \det(A) = 0.$$

$\therefore$ A is singular iff $\lambda = 0$ is an eigenvalue of A ▨

Q4) Let A be a nonsingular matrix and let λ be an eigenvalue of A. Show that $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Solution. Since A is a nonsingular matrix and λ is an eigenvalue of A, then there exists a vector $v \neq 0$ such that

$$Av = \lambda v.$$

$\Rightarrow A^{-1}AV = \lambda A^{-1}V \Rightarrow V = \lambda A^{-1}V$

It follows from Exercise 3 that $\lambda \neq 0$.

$\Rightarrow A^{-1}V = \frac{1}{\lambda}V, \; V \neq 0$.

and hence $\frac{1}{\lambda}$ is an eigenvalue of A^{-1} ⊟

Q8) Let A be an $n \times n$ matrix such that $A^2 = A$. Show that if λ is an eigenvalue of A, then $\lambda = 0$ or $\lambda = 1$.

Solution. Since λ is an eigenvalue of A, then there exists a vector $V \neq 0$ such that

$$AV = \lambda V,$$

$$\Rightarrow A(AV) = \lambda AV$$

$$\boxed{A^2V = \lambda(\lambda V) = \lambda^2 V} \text{ ----(i)}$$

Since $A^2 = A$, then $\boxed{A^2V = AV = \lambda V}$ ---(ii)

(i) − (ii) : $\vec{0} = A^2V - A^2V = (\lambda^2 - \lambda)\vec{V}, \; \vec{V} \neq 0$

$\Rightarrow \lambda^2 - \lambda = 0 \Rightarrow \boxed{\lambda = 0}$ or $\boxed{\lambda = 1}$

Q14) Let A be a 2×2 matrix. If $tr(A)=8$ and $|A|=12$, what are the eigenvalues of A.

Solution, $\det(A-\lambda I)=0$, $A_{2\times2}$.

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda) - a_{21}a_{12} = 0$$

$$\Rightarrow \lambda^2 - (\underbrace{a_{11}+a_{22}}) \lambda + (\underbrace{a_{11}a_{22}-a_{21}a_{12}}) = 0$$

$$\Rightarrow \boxed{\lambda^2 - tr(A)\lambda + \det(A) = 0} \quad (*)$$

$$\Rightarrow \lambda^2 - 8\lambda + 12 = 0$$

$$(\lambda-6)(\lambda-2)=0$$

$\lambda_1 = 6$, $\lambda_2 = 2$ are the eigenvalues ∎

Q16) Let A be a 2×2 matrix and let

$(**)$ $\boxed{P(\lambda) = \lambda^2 + b\lambda + c}$ be the characterstic polynomial of A. Show that

$$b = -tr(A), \quad c = \det(A).$$

Solution See Q14, comparing $(*)$ + $(**)$, we get $b = -tr(A)$, $c = \det(A)$ ∎

Section 6.3. Q6, Q9.

Q6) Let A be a diagonalizable matrix whose eigenvalues are either 1 or -1. show that $A^{-1} = A$.

Solution. Since A is diagonalizable, then $A = XDX^{-1}$, where D is a diagonal matrix whose diagonal elements are all either 1 or -1, then $D^{-1} = D$ and

$$A^{-1} = (XDX^{-1})^{-1} = (X^{-1})^{-1} D^{-1} X^{-1}$$

$$= XDX^{-1} =$$ ▤

Q9) Let A be a 4×4 matrix and let λ be an eigenvalue of multiplicity 3. If $A - \lambda I$ has rank 1, is A defective? Explain.

Solution. Rank-nullity then gives

$$\underbrace{\text{Rank}(A-\lambda I)}_{1} + \text{nullity}(A-\lambda I) = 4$$

$$\Rightarrow \text{nullity}(A-\lambda I) = \dim N(A-\lambda I) = 3.$$

Since λ has multiplicity 3, then the matrix A is NOT defective (diagonalizable) ▤

5.4 Inner Product spaces

Df. An inner product on a vector space V is an operation on V that assigns, to each pair of vectors x and y in V, a real number $<x, y>$ satisfying the following conditions.

(i) $<x,x> \geq 0$ with equality iff $x=0$.

(ii) $<x,y> = <y,x>$, $\forall\ x,y \in V$.

(iii) $<\alpha x+\beta y, z> = \alpha <x,z> + \beta <y,z>$ $\forall\ x,y,z$ in V and all scalars α and β

- A vector space V with an inner product is called an inner product space.

Examples: ① The vector space $\mathbb{R}^n$:

The standard inner product for $\mathbb{R}^n$ is

$$\boxed{<x,y> = x^T y}$$

That is, $\boxed{\left\langle \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = \sum_{i=1}^{n} x_i y_i}$ $(*)$

- We also define an inner product on $\mathbb{R}^n$ by $\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i w_i$, where w_i are referred to as <u>weights</u>.

. verification of $(*)$ as inner product.

(i) $\langle x, x\rangle = \left\langle \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \right\rangle = \sum_{i=1}^{n} x_i x_i$

$$= \sum_{i=1}^{n} x_i^2 \geq 0$$

and $\langle x, x\rangle = 0$ iff $\sum_{i=1}^{n} x_i^2 = 0$

iff $x_1 = x_2 = \cdots = x_n = 0$

iff $x = 0$.

(ii) $\langle x, y\rangle = \left\langle \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \right\rangle = \sum_{i=1}^{n} x_i y_i$

$$= \sum_{i=1}^{n} y_i x_i = \langle y, x\rangle.$$

(iii) $\langle \alpha x + \beta y, z\rangle = \left\langle \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \vdots \\ \alpha x_n + \beta y_n \end{pmatrix}, \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \right\rangle$

$$= (\alpha x_1 + \beta y_1) z_1 + \cdots + (\alpha x_n + \beta y_n) z_n$$

$$= \alpha x_1 z_1 + \beta y_1 z_1 + \cdots + \alpha x_n z_n + \beta y_n z_n$$

$$= \alpha (x_1 z_1 + \cdots + x_n z_n) + \beta (y_1 z_1 + \cdots + y_n z_n)$$

$$= \alpha \langle x, z \rangle + \beta \langle y, z \rangle ,$$

for all $x, y, z \in \mathbb{R}^n$, and α, β scalars.

<u>Ex2.</u> The vector space $\mathbb{R}^{m \times n}$.

Given A and B in $\mathbb{R}^{m \times n}$, we can define an inner product by

$$\langle A, B \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij} .$$

<u>Ex.</u> If $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 3 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 1 \\ 3 & 0 \\ -3 & 4 \end{bmatrix}$

$$\langle A, B \rangle = (1)(-1) + (1)(1) + (1)(3) + (2)(0) + (3)(-3) + (3)(4) = 6$$

Ex3: The vector space $C[a,b]$.

The inner product on $C[a,b]$ defined by $\boxed{\langle f,g\rangle = \int_a^b f(x)g(x)\,dx}$.

proof. (i) $\langle f,f\rangle = \int_a^b (f(x))^2\,dx \geq 0$.

$$\langle f,f\rangle = 0 \Leftrightarrow \int_a^b f^2(x)\,dx = 0$$
$$\Leftrightarrow f \equiv 0 \text{ on } [a,b]$$ (why?

(ii) $$\langle f,g\rangle = \int_a^b f(x)g(x)\,dx = \int_a^b g(x)f(x)\,dx = \langle g,f\rangle.$$

(iii) $$\langle \alpha f+\beta g, h\rangle = \int_a^b (\alpha f(x)+\beta g(x))\,h(x)\,dx$$
$$= \alpha\int_a^b f(x)h(x)\,dx + \beta\int_a^b g(x)h(x)\,dx$$
$$= \alpha\langle f,h\rangle + \beta\langle g,h\rangle,$$

$\forall f,g,h \in C[a,b]$, $\forall \alpha,\beta$ scalars.

If $w(x)$ is a positive continuous function on $[a,b]$, then

$$\boxed{\langle f,g\rangle = \int_a^b f(x)g(x)w(x)\,dx}$$ defines

an inner product on $C[a,b]$.

the function $w(x)$ is called a <u>weight function</u>

<u>Ex 4.</u> the vector space P_n.

Let $x_1, x_2, \ldots, x_n$ be distinct real numbers.

$\forall\, p, q \in P_n$, we define an inner product on P_n by

$$\langle p,q\rangle = \sum_{i=1}^{n} p(x_i)\, q(x_i).$$

<u>proof.</u> (i) $\langle p,p\rangle = \sum_{i=1}^{n} p^2(x_i) \geq 0$.

$\langle p,p\rangle = 0 \quad \sum_{i=1}^{n} p^2(x_i) = 0$

$\Rightarrow p(x_i) = 0 \quad \forall\, i = 1, \ldots, n$

$\Rightarrow x_1, \ldots, x_n$ must be roots of $p(x)$.

Since $p(x)$ is of degree less than n, it must be the zero polynomial.

(ii) $$\langle p, q\rangle = \sum_{i=1}^{n} p(x_i)\, q(x_i)$$

$$= \sum_{i=1}^{n} q(x_i)\, p(x_i) = \langle q, p\rangle .$$

(iii) $$\langle \alpha p+\beta q, h\rangle = \sum_{i=1}^{n} (\alpha p+\beta q)(x_i)\, h(x_i)$$

$$= \alpha \sum_{i=1}^{n} p(x_i) h(x_i) + \beta \sum_{i=1}^{n} q(x_i) h(x_i)$$

$$= \alpha \langle p, h\rangle + \beta \langle q, h\rangle .$$

∎

- If $w(x)$ is a positive function, then

$$\boxed{\langle p, q\rangle = \sum_{i=1}^{n} p(x_i)\, q(x_i)\, w(x_i)}$$ also defines an inner product on P_n.

Basic Properties of Inner Product Spaces

- If V is a vector in an inner product space V, then the length, or norm of V is given by

$$||V|| = \sqrt{\langle V,V \rangle}.$$

- Two vectors u and V are said to be orthogonal if $\langle u, V \rangle = 0$

(The Pythagorean Law)

Thm: If u and V are orthogonal vectors in an inner product space V, then $||u+V||^2 = ||u||^2 + ||V||^2$.

Proof:
$$\begin{aligned} ||u+v||^2 &= \langle u+v, u+v \rangle \\ &= \langle u,u \rangle + 2\langle u,V \rangle + \langle V,V \rangle \\ &= ||u||^2 + 0 + ||V||^2 \\ &= ||u||^2 + ||V||^2. \end{aligned}$$

∎

Ex 5. Consider the vector space $C[-1,1]$. then 1 and x are orthogonal.

Sol: $\langle 1, x \rangle = \int_{-1}^{1} 1 \cdot x \, dx = \frac{x^2}{2} \Big|_{-1}^{1} = 0.$

- the length of 1 is

$$\|1\| = \sqrt{\langle 1, 1 \rangle} = \sqrt{\int_{-1}^{1} (1 \cdot 1) \, dx} = \sqrt{2}$$

- The length of x is

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\int_{-1}^{1} x \cdot x \, dx} = \sqrt{\frac{x^3}{3} \Big|_{-1}^{1}}$$

$$= \sqrt{\frac{2}{3}} = \frac{\sqrt{6}}{3}$$

Since 1 and x are orthogonal, then they satisfy Pythagorean law:

$$\|1 + x\|^2 = \|1\|^2 + \|x\|^2 = 2 + \frac{2}{3} = \frac{8}{3}.$$

OR $\|1+x\|^2 = \langle 1+x, 1+x \rangle = \int_{-1}^{1} (1+x)^2 dx = \frac{8}{3}.$

Ex 6. For the vector space $C[-\pi,\pi]$, we use a constant weight function $w(x) =$

then the inner product defined by

$$\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,g(x)\,dx \quad , \forall f,g \in C[-\pi,\pi$$

then $\langle \cos x, \sin x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos x \sin x\,dx = 0$

$$\langle \cos x, \cos x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos^2 x\,dx = 1$$

$$\langle \sin x, \sin x\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2 x\,dx = 1.$$

Ex 7. For the vector space $\mathbb{R}^{m\times n}$, the norm (Frobenius norm) is given by

$$\|A\|_F = \sqrt{\langle A,A\rangle} = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2\right)^{1/2}$$

Ex. $A = \begin{bmatrix} -1 & 1 \\ 3 & 0 \\ -3 & 4 \end{bmatrix}$, $\|A\|_F = (1+1+9+0+9+16)^{\frac{1}{2}}$

$= 6.$

Ex7. In P_5, define an inner product by $\langle p,q\rangle = \sum_{i=1}^{n} p(x_i)\, q(x_i)$, where $x_1, x_2, \dots, x_n$ are distinct real numbers.

let $x_i = \frac{i-1}{4}$, $i=1,2,\dots,5$

That is, $x_1 = 0$, $x_2 = \frac{1}{4}$, $x_3 = \frac{1}{2}$, $x_4 = \frac{3}{4}$, $x_5 = 1$

find the length of $p(x) = 4x$.

Sol. $\|4x\| = \sqrt{\langle 4x, 4x\rangle} = \left(\sum_{i=1}^{5} 16 x_i^2\right)^{\frac{1}{2}}$

$$= \left[\sum_{i=1}^{5} \cancel{16} \frac{(i-1)^2}{\cancel{16}}\right]^{\frac{1}{2}}$$

$$= (0+1+4+9+16)^{\frac{1}{2}}$$

$$= \sqrt{30}.$$

thm: (The Cauchy - Schwarz inequality)

If u and v are any vectors in an inner product space V, then $\boxed{|\langle u,v\rangle| \leq \|u\|\,\|v\|}$

$|\langle u,v\rangle| = \|u\|\,\|v\|$ iff u and v are linearly dependent.

Norms.

Df. A vector space V is said to be a normed linear space if, to each vector $v \in V$, there is associated a real number $\|v\|$, called the norm of v, satisfying

(I) $\|v\| \geq 0$ with equality iff $v=0$.

(II) $\|\alpha v\| = |\alpha| \|v\|$ for any scalar α.

(III) $\|v+w\| \leq \|v\| + \|w\|$, $\forall v, w \in V$ (this is called the triangle inequality).

Thm: If V is an inner product space, then the equation $\|v\| = \sqrt{<v,v>}$, $\forall v \in V$ defines a norm on V.

proof. (Exercise).

Rmk. It is possible to define many different norms on a given vector space.

Ex. 8 In $\mathbb{R}^n$, we could define

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| , \forall x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n.$$

Show that $\|\cdot\|_1$ defines a norm on $\mathbb{R}^n$.

Proof. (I) $\|x\|_1 = \sum_{i=1}^{n} |x_i| \geq 0$

$$\|x\|_1 = 0 \text{ iff } |x_i| = 0 , \forall i=1,\dots,n$$
$$\text{iff } x_i = 0 , i=1,\dots,n$$
$$\text{iff } x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 0 .$$

(II) $$\|\alpha x\|_1 = \sum_{i=1}^{n} |\alpha x_i| = |\alpha| \sum_{i=1}^{n} |x_i|$$
$$= |\alpha| \, \|x\|_1 .$$

(III) $$\|x+y\|_1 = \sum_{i=1}^{n} |x_i + y_i|$$
$$\leq \sum_{i=1}^{n} |x_i| + \sum_{i=1}^{n} |y_i|$$
$$= \|x\|_1 + \|y\|_1 . \; ▨$$

Ex 9 $||x||_\infty = \max_{1 \le i \le n} |x_i|$ defines a norm on $\mathbb{R}^n$.

Proof. (Exercise).

Ex 10 $||x||_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$, for any real number $p \geq 1$ defines a norm on $\mathbb{R}^n$. In particular, if $p = 2$, then $||x||_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}} = \sqrt{<x,x>}$

Ex 11. Let $x = \begin{pmatrix} 4 \\ -5 \\ 3 \end{pmatrix} \in \mathbb{R}^3$. Compute $||x||_1$, $||x||_2$, $||x||_\infty$.

Sol. $||x||_1 = \sum_{i=1}^{3} |x_i| = |4| + |-5| + |3| = 12$

$$||x||_2 = \left(\sum_{i=1}^{3} |x_i|^2\right)^{\frac{1}{2}} = \left(|4|^2 + |-5|^2 + |3|^2\right)^{\frac{1}{2}}$$

$$= (16 + 25 + 9)^{1/2} = 5\sqrt{2}$$

$$||x||_\infty = \max_{1 \le i \le 3} |x_i| = \max\{|4|, |-5|, |3|\}$$

$$= 5 .$$

Recording#1 (4/6/2020)
https://us02web.zoom.us/rec/share/v8Z_f6zbq35OY5HRuWHncPYLLJ31aaa81CJL-PJexBrg2mcxmTjK7UKpeIjNWTHT
Password: 0f+l@58&
https://drive.google.com/open?id=1m8VGdqnwywHIrPILOmVuR6Dk45NgrQJx

Recording#2 (6/6/2020)
https://us02web.zoom.us/rec/share/uFUF5WqrElOWLeKtHjwSLcCP6reT6a8gHVIqfEInkwhiqAFM6X1Rv7RxAMoMKKF

Password: 4f#2%wjI

https://drive.google.com/open?id=1O1SA0R3FZaO_6HFG4Cs88rgOBQezxuQ9

Recording#3 (8/6/2020)

https://us02web.zoom.us/rec/share/uONycIut_CRIU7fM-WHvQZJwP6D6T6a80CZP__QEyEoKb9gABH2FQwMVoyXC1OpC

Password: 6m%52i$4

https://drive.google.com/open?id=102Lg4tA5lv2XWNXV46deCZdG-hm_u9hq

Recording#4 (9/6/2020)

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Recording#5 (10/6/2020)

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Recording#6 (11/6/2020)

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Recording#7 (13/6/2020)

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Recording#8 (15/6/2020)

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Password: 0L%+mo00

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Recording#9 (16/6/2020)

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Recording#10 (17/6/2020)

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Recording#11 (18/6/2020)

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Recording#12 (20/6/2020)

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Recording#13 (22/6/2020)

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Password: 6x$lZ6R*

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Recording#14 (23/6/2020)

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Password: 9k*I745m.

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Recording#15 (24/6/2020)

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Password: 1q*$&f8*

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Recording#16 (25/6/2020)

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Password: 6V^66=^.

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Recording# 17(27/6/2020)

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Recording# 18(29/6/2020)

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Recording#19 (30/6/2020)

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Recording#20 (01/07/2020)

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Recording#21 (02/07/2020)

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Recording#22 (06/07/2020)

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Recording#23 (07/07/2020)

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Password: 4Y&+C+4?

https://drive.google.com/file/d/1B99PkXLbh1jQ2-H0tbFmlocs24UUIQrr/view?usp=drivesdk

Birzeit University
Mathematics Department
Math234
Quiz#1

Instructor: Dr. Ala Talahmeh **First Summer Semester 2020**
Name:.. **Number:**..............
Section: (3) **Date: 13/06/2020**

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1. Any $m \times n$ linear system $Ax = 0$ has a nontrivial solution if $m < n$.

   A) True
   B) False

2. If $[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 2 & -1 & 2 & 6 \\ 1 & 1 & 2 & 5 \end{array}\right]$. is the augmented matrix of the system $Ax = b$ then the system has infinitely many solution.

   A) True
   B) False

3. Consider a linear system whose augmented matrix is $[A|b] = \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 0 \\ 2 & 3 & 1 & 1 & -1 \\ 0 & 1 & 1 & \alpha & \beta \end{array}\right]$. The system has infinitely many solution if

   A) $\alpha \neq -3$ and β any number or $\alpha = -3$ and $\beta = 1$
   B) $\alpha = -3$ and $\beta \neq 1$
   C) Not possible
   D) $\alpha = -3$ and $\beta = 1$

4. The above system is inconsistent if

   A) $\alpha \neq -3$ and β any number or $\alpha = -3$ and $\beta = 1$
   B) $\alpha = -3$ and $\beta \neq 1$
   C) Not possible
   D) $\alpha = -3$ and $\beta = 1$

5. Consider a linear system whose augmented matrix is $[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & \alpha^2 - 14 & \alpha + 2 \end{array}\right]$. The system has infinitely many solution if

   A) $\alpha \neq \pm 4$
   B) $\alpha = 4$
   C) Not possible
   D) $\alpha = -4$

Good Luck

Quiz #1 (key)

1) Since it is undetermined homogeneous System

2) $$[A|b] = \left[\begin{array}{ccc|c} 1 & 1 & 2 & 4 \\ 2 & -1 & 2 & 6 \\ 0 & 0 & 0 & 1 \end{array}\right] \text{ is inconsistent.}$$

$-R_1+R_3$

3) $$[A|b] = \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 0 \\ 2 & 3 & 1 & 1 & -1 \\ 0 & 1 & 1 & \alpha & \beta \end{array}\right]$$

$$\xrightarrow[-2R_1+R_2]{} \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 1 & 1 & \alpha & \beta \end{array}\right]$$

$$\xrightarrow[R_2+R_3]{} \left[\begin{array}{cccc|c} 1 & 2 & 1 & -1 & 0 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & 0 & \alpha+3 & \beta-1 \end{array}\right]$$

Notice that this system is underdetermined.
the system has infinitely many solutions
if $\alpha=-3$ and $\beta=1$ $\boxed{\text{OR}}$ $\alpha\neq-3$ and $\beta\in\mathbb{R}$.

4) The above system is inconsistent if
$\alpha=-3$, $\beta\neq1$

5) $[A|b] = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & \alpha^2-14 & \alpha+2 \end{array}\right]$

$$\xrightarrow[\substack{-3R_1+R_2 \\ -4R_1+R_3}]{} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & \alpha^2-2 & \alpha-14 \end{array}\right]$$

$$\xrightarrow[-R_2+R_3]{} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & \alpha^2-16 & \alpha-4 \end{array}\right]$$

The system has infinitely many solution

if $\alpha = 4$

The system has aunique solution

if $\alpha \neq \pm 4$.

The system has no solution

if $\alpha = -4$

Birzeit University
Mathematics Department
Math234
Quiz#2

Instructor: Dr. Ala Talahmeh | **First Summer Semester 2020**
Name:... | **Number:**...............
Section:(3) | **Date: 18/06/2020**

Question I [50 marks]. True or **False**.

1. The sum of two elementary matrices is an elementary matrix. F
2. If A is a nonsingular matrix, then A^T is nonsingular. T
3. If A is a singular matrix, then the system $Ax = 0$ has a unique solution. F
4. If A is symmetric and nonsingular, then A^{-1} is symmetric. T
5. If A is an $n \times n$ nonsingular matrix, then A^{100} is nonsingular.T
6. If A and B are 6×6 symmetric matrices, then $AB - BA$ is skew-symmetric. T
7. If A is an $n \times n$ matric such that $AA^T = I$, then $A^T = A^{-1}$. T
8. If A is an $m \times n$ matrix such that $AA^T = O$ or $A^TA = O$, then $A = O$. T
9. If b is any column of the matrix A, then the system $Ax = b$ is consistent. T
10. The vector $(1, 0, 0)^T$ is a linear combination of the vectors $(1, 2, 3)^T, (1, 4, 1)^T; (2, 3, 1)^T$. T
11. If A, B are square $n \times n$ matrices such that $AB = O$, then A or B is nonsingular. F
12. If $A = LU$ is the $LU-$factorization. Then U is singular iff A is singular. T
13. If E is an elementary matrix, then $E + E^T$ is an elementary matrix. F
14. If A and B are $n \times n$ matrices such that $Ax = Bx$ for some none zero $x \in \mathbb{R}^n$. Then $A-B$ is nonsingular. F
15. Every square matrix can be expressed as the sum of a symmetric matrix and a skew-symmetric matrix. T
16. If $A^TA = A$, then A is symmetric and $A^2 = A$. T
17. The sum of a symmetric and skew-symmetric matrices is symmetric. F
18. Let A be nonsingular. If A is skew-symmetric, then A^{-1} is skew-symmetric. T
19. If A is a 3×3 matrix and $(2, 3, -1)$ is a solution to $Ax = 0$, then $(8, 12, -4)$ is also a solution. T
20. If A is an $n \times n$ nonsingular matrix, then the system $Ax = b$ has infinitely many solutions. F
21. If A is a 4×4 nonsingular matrix, then AA^T is both symmetric and nonsingular. T
22. If A is a 4×4 matrix and $Ax = 0$ has only the zero solution, then A is row equivalent to I. T
23. A square matrix A is nonsingular iff its reduced row echelon form is the identity matrix. T
24. If A, B, C are $n \times n$ matrices such that $AB = AC$, then $B = C$. F
25. In the linear system $AX = b$, if $b = a_1 - a_2 + 3a_4,\ a_1 = -a_3$, then the system has infinite solutions. T

Good Luck

Birzeit University
Mathematics Department
Math234
First Exam (Take Home)

Instructor: Dr. Ala Talahmeh **First Summer Semester 2019/2020**
Name:.. **Date: 21/06/2020**

Exercise#1 [5 marks]. Use the **Gauss-Jordan reduction method** to solve the following system:

$$x_1 - x_2 + 2x_3 + x_4 = -1$$
$$-2x_1 + x_3 - x_4 = 3$$
$$3x_1 + x_2 - x_3 = 1$$
$$2x_1 - 4x_2 + 9x_3 + 3x_4 = -1$$

Exercise#2 [5 marks]. Consider the linear system

$$x_1 + x_2 + 2x_3 = 1$$
$$x_1 + \alpha x_2 - x_3 = 2$$
$$2x_1 - x_2 + \beta x_3 = -1$$

For which values of α and β will the linear system have a **unique solution**?

Exercise#3 [5 marks]. Using **Cramer's rule,** find only the value of the angle β such that

$$2\sin\alpha - \cos\beta + 3\tan\gamma = 3$$
$$4\sin\alpha + 2\cos\beta - 2\tan\gamma = 2$$
$$6\sin\alpha - 3\cos\beta + \tan\gamma = 9,$$

where $0 \le \beta \le 2\pi$.

Exercise#4 [11 marks]. Let $N = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$.

(a) Find N^n for every $n \ge 3$.

(b) Use **part (a)** to find A^n for every $n \ge 2$ [**Hint:** Write $A = N + I$].

Exercise#5 [10 marks]. Let $A = \begin{pmatrix} 2 & 4 & 1 \\ 4 & 3 & 4 \\ 1 & 4 & 2 \end{pmatrix}$.

(a) Find elementary matrices E_1, E_2, E_3 so that $E_3E_2E_1A = U$, where U is an upper triangular matrix.

(b) Use **part (a)** to find an **LU-factorization** of the matrix A.

Exercise#6 [10 marks].

(a) Find X in the matrix equation:

$$X = X\begin{pmatrix} 0 & -2 & -1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

(b) Find $[\mathrm{adj}(A)]^{-1}$ given that $A^{-1} = \begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$.

Exercise#7 [24 marks]. Prove or **disprove.**

1. Let A be an $n \times n$ nonsingular matrix. If $\det\big(\mathrm{adj}A\big) = \det(A)$, then A is 2×2 matrix.
2. If A and B be $n \times n$ symmetric matrices, then $AB = BA$ if and only if AB is also symmetric.
3. A triangular matrix is nonsingular if and only if its diagonal elements are all nonzero.
4. If A is an $n \times n$ matrix, then $A = M - N$, where M is symmetric and N is skew symmetric.
5. If A is an $n \times n$ matrix such that $A^2 - 4A + 4I_n = O_n$, where I_n is the $n \times n$ identity matrix and O_n is the $n \times n$ zero matrix, then A is invertible and $A^{-1} = I_n - \frac{1}{4}A$.
6. If x and y are two distinct vector in $\mathbb{R}^n$ such that $Ax = Ay$, then $\det(A) = 0$.
7. If A and B are $n \times n$ matrices, then $\det((AB)^T) = \det(A)\det(B)$.
8. If A, B, C are $n \times n$ matrices such that C is nonsingular and $A = CBC^{-1}$, then $\det(A) = \det(B)$.

Good Luck

Birzeit University
Mathematics Department
Math234
Second Exam (KEY)

Instructor: Dr. Ala Talahmeh **First Summer Semester 2020**
Name:.. **Number:**..............
Section:(3) **Date: 04/07/2020**

Exercise 1 [**40 marks**]. Answer by **true** or **false**.

1. **(F)** If the set $\{v_1, v_2, ..., v_k\}$ spans P_4, then $k = 4$.
2. **(F)** If u, v, w are nonzero vectors in $\mathbb{R}^2$, then $w \in span(u, v)$.
3. **(T)** If A is a 4×4 matrix with $a_2 + a_4 = 0$,, then $N(A) \neq \{0\}$.
4. **(T)** If the vectors u_1, u_2, u_3, u_4 span $\mathbb{R}^{2\times 2}$, then they are linearly independent.
5. **(F)** The coordinate vector of $q(x) = 4 + 6x$ with respect to the basis $[2x, 2]$ is $(2, 3)^T$.
6. **(T)** The transition matrix of two basis is nonsingular.
7. **(T)** If $dim\ V = n < +\infty$, then an n linearly independent set of vectors in V is a basis for V.
8. **(T)** Let $W = \{(x, y, x + y + 2z)^T : x, y, z \in \mathbb{R}\}$, then $\{(1, 0, 1)^T, (0, 0, 1)^T, (0, 1, 1)^T\}$ is a basis for W.
9. **(T)** Let $S = Span\ \{v_1, v_2, v_3, v_4\}$ and suppose that $v_1 = v_2 + v_3,\ v_3 = v_2 - v_4$, then $S = Span\ \{v_3, v_4\}$.
10. **(F)** Let $S = \{ax^2 + ax : a \in \mathbb{R}\}$, then $\{x^2, x\}$ is a basis for S.
11. **(T)** If $f_1, ...f_n$ are linearly dependent, then Wronskian($f_1, ...f_n$)=0.
12. **(F)** The set $S = \{(x, y) : y = x + 3\}$ is a subspace of $\mathbb{R}^2$.
13. **(T)** The dimension of the subspace $W = \{A \in \mathbb{R}^{2\times 2} :\ A\ is\ symmetric\}$ is 3.
14. **(F)** If V is a vector space with $dim(V) = 4$ and $\{v_1, v_2, v_3, v_4\} \subseteq V$, then $span\{v_1, v_2, v_3, v_4\} = V$.
15. **(F)** If A is a singular $n \times n$ matrix, then $rank(A) = n$.
16. **(T)** If S is a subspace of a vector space V, then S is a vector space.
17. **(T)** If $\{v_1, v_2, v_3\}$ are vectors in a vector space V and $Span\ \{v_1, v_2\} = Span\ \{v_1, v_2, v_3\}$, then $\{v_1, v_2, v_3\}$ are linearly dependent.
18. **(F)** If A is a 3×3 matrix and $rank(A) = 2$, then A is nonsingular.
19. **(T)** If A is an $m \times n$ matrix, then A and A^T have the same rank.
20. **(T)** If A is an 3×4 matrix, then $rank(A) \leq 3$.

Exercise 2 [**20 marks**]. Circle the correct answer.

1. let $S = \{p \in P_3 : \; p(0) = 0\}$. One of the following is a basis for S.
   - **(a)** $\{1, x, x^2\}$
   - [✓ (b)] $\{x, x^2\}$
   - **(c)** $\{x^2 + x\}$
   - **(d)** $\{x^2 + 1\}$

2. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then
   - **(a)** A is in REF
   - **(b)** nullity(A)=2
   - [✓ (c)] rank(A)=2
   - **(d)** $\det(A) \neq 0$

3. The vectors $\{2, x, \sin x\}$ in $C[0, 2\pi]$ are
   - [✓ (a)] Linearly independent
   - **(b)** Linearly dependent
   - **(c)** A basis for $C[0, 2\pi]$
   - **(d)** A spanning set for $C[0, 2\pi]$

4. Consider the ordered basis $E = \{e_1, e_1 - e_2\}$ for $\mathbb{R}^2$. If $[v]_E = (1, -1)^T$, then $v =$
   - **(a)** $-e_2$
   - **(b)** $2e_1 + e_2$
   - **(c)** $e_1 + e_2$
   - [✓ (d)] $(0, 1)^T$

5. If the reduced row echelon form of A is $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ and $a_2 = (2, 2)^T$, then $A =$
   - **(a)** $\begin{bmatrix} 4 & 2 & 2 \\ 4 & 2 & 2 \end{bmatrix}$
   - **(b)** $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$
   - [✓ (c)] $\begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \end{bmatrix}$
   - **(d)** $\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

6. The transition matrix from the basis $E = [1, -x]$ to the basis $F = [-1, x-1]$ of P_2 is

(a) $\begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

7. If $A = \begin{bmatrix} 1 & -3 & 2 & 3 \\ -6 & 6 & -4 & -5 \end{bmatrix}$, then

(a) rank(A)=1, nullity(A)=3.

(b) rank(A)=3, nullity(A)=1.

(c) rank(A)=4, nullity(A)=0.

(d) rank(A)=nullity(A)=2.

8. The dimension of the vector space spanned by $\{1-x-x^2,\ 1+x+x^2,\ 2-x,\ 2x-4\}$ is

(a) 1

(b) 2

(c) 3

(d) 4

9. One of the following sets is a subspace of P_4

(a) $\{f(x) \in P_4 : f(0) = 1\}$

(b) $\{f(x) \in P_4 : f(1) = 1\}$

(c) $\{f(x) \in P_4 : f(1) = 0\}$

(d) $\{f(x) \in P_4 : f(0) = 0,\ f'''(0) = 6\}$

10. If A is a 4×3 matrix such that $N(A) = \{0\}$, and $b = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix}$, then

(a) It is possible that $Ax = b$ has infinitely many solutions

(b) The system $Ax = b$ has exactly one solution.

(c) The system $Ax = b$ has at most one solution.

(d) The system $Ax = b$ has no solution

Exercise 3 [**8 marks**]. Let $V = \mathbb{R}^{2\times 2}$ be the vector space of all 2×2 matrices. Let W_1 be the set of matrices of the form $\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$, and W_2 set of matrices of the form $\begin{bmatrix} a & b \\ -a & c \end{bmatrix}$.

a. Find $W_1 \cap W_2$.

b. Find a basis and dimention of $W_1 \cap W_2$.

Exercise 4 [**12 marks**]. Let V be the vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$; let V_e be the subset of even functions and V_o be the subset of odd functions.

a. Prove that V_e and V_o are subspaces of V. **(Do only one case)**.

b. Prove that $V_e \cap V_o = \{0\}$.

c. Prove that $V = V_e + V_o$.

Good Luck

Exercise #3. $V_e = \{ f : \mathbb{R} \to \mathbb{R} : f(-x) = f(x) \}$

$V_o = \{ f : \mathbb{R} \to \mathbb{R} : f(-x) = -f(x) \}$

(a) V_e is a subspace (V_o is similar).

(i) $0(-x) = 0(x) \Rightarrow 0 \in V_e \quad \therefore V_e \neq \Phi$.

(ii) let $f, g \in V_e$. then $f(-x) = f(x)$, $g(-x) = g(x)$.

$$(f+g)(-x) = f(-x) + g(-x)$$
$$= f(x) + g(x)$$
$$= (f+g)(x).$$

$\therefore f+g \in V_e$.

(iii) $\forall f \in V_e$, $\alpha \in \mathbb{R}$, we have

$$(\alpha f)(-x) = \alpha f(-x)$$
$$= \alpha f(x) \quad \text{since } f \in V_e$$
$$= (\alpha f)(x)$$

$\therefore \alpha f \in V_e$.

(b) $V_e \cap V_o = \{0\}$.

let $f \in V_e \cap V_o$, then $f \in V_e$ and $f \in V_o$

$\Rightarrow f(-x) = f(x)$ and $f(-x) = -f(x)$

$\Rightarrow 2f(x) = 0 \Rightarrow f(x) = 0$

(c) $V = V_e + V_o$

let $f \in V$, then $f(x) = \underbrace{\left(\frac{f(x)+f(-x)}{2}\right)}_{\in V_e} + \underbrace{\left(\frac{f(x)-f(-x)}{2}\right)}_{\in V_o}$

Exercise #4. $V = \mathbb{R}^{2\times 2}$

$$W_1 = \left\{ A \in \mathbb{R}^{2\times2} : A = \begin{bmatrix} x & -x \\ y & z \end{bmatrix} \right\}$$

$$W_2 = \left\{ B \in \mathbb{R}^{2\times2} : B = \begin{bmatrix} a & b \\ -a & c \end{bmatrix} \right\}$$

a) $W_1 \cap W_2 = \left\{ C \in \mathbb{R}^{2\times2} : C = \begin{bmatrix} \alpha & -\alpha \\ -\alpha & \beta \end{bmatrix} \right\}$

b) $W_1 \cap W_2 = \left\{ C \in \mathbb{R}^{2\times2} : C = \alpha \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} + \beta \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

$$= \text{Span}\left\{ \underbrace{\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}_{\text{lin. indep check!}} \right\}$$

A basis for $W_1 \cap W_2$ is

$$\left\{ \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$\dim W_1 \cap W_2 = 2$