8.2 Trigonometric Integrals

Note Title ٢٣/٠٤/١٣

إنه من أهم ما يميز الدوال المثلثية الستة هو وجود المتطابقات المثلثية والتي كثيراً ما تستخدم لتسهيل التعامل مع مثل هذه الدوال / مثال ذلك من التكاملات يمكننا استخدام العلاقة $\tan^2 x = \sec^2 x - 1$ لحل التكامل التالي

$$\int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx = \tan x - x + C$$

في هذا الفصل سنقوم بالتعرف على أفكار إضافية للتعامل مع التكاملات التي تحتوي الدوال المثلثية.

Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the appropriate substitution into three cases according to m and n being odd or even.

Case 1 If **m is odd**, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \qquad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If **m is even and n is odd** in $\int \sin^m x \cos^n x \, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If **both m and n are even** in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \qquad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Example: Evaluate 1) $\int \sin^3 x \cos^2 x \, dx$

<u>Sol:</u> $\int \sin^2 x \cos^2 x \cdot \sin x \, dx = \int (1-\cos^2 x)\cos^2 x \cdot \sin x \, dx$

$$= \int (\cos^2 x - \cos^4 x)\sin x \, dx \qquad u = \cos x$$

$$= -\int (u^2 - u^4)\, du = \frac{u^5}{5} - \frac{u^3}{3} + C \qquad du = -\sin x \, dx$$

$$-du = \sin x \, dx$$

$$= \boxed{\frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C}$$

2) $\int \cos^5 x \, dx$

بداية يمكن حلها باستخدام فكرة صيغ الاختزال كما في الفصل السابع من خلال التكامل بالأجزاء

$$\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx = \int (1-\sin^2 x)^2 \cos x \, dx$$

$$= \int (1 - 2\sin^2 x + \sin^4 x)\cos x \, dx \qquad u = \sin x$$

$$= \int (1 - 2u^2 + u^4)\, du = u - \frac{2}{3}u^3 + \frac{u^5}{5} + C \qquad du = \cos x \, dx$$

$$= \boxed{\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C}$$

3) $\int \sin^2 x \cos^4 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx$

$$= \frac{1}{8}\int (1-\cos^2 2x)(1+\cos 2x)\, dx = \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx$$

$$= \frac{1}{8}\left[x + \frac{\sin 2x}{2} - \int \frac{1+\cos 4x}{2}\, dx - \int (1-\sin^2 2x)\cos 2x \, dx\right]$$

$$= \frac{x}{8} + \frac{\sin 2x}{16} - \frac{1}{16}\left[x + \frac{\sin 4x}{4}\right] - \frac{1}{8}\int (1-u^2)\frac{du}{2} \qquad u = \sin 2x, \; du = (\cos 2x)\, 2\, dx$$

$$= \frac{x}{8} + \frac{\sin 2x}{16} - \frac{1}{16}\left[x + \frac{\sin 4x}{4}\right] - \frac{1}{16}\left[\sin 2x - \frac{\sin^3 2x}{3}\right] + C$$

$$= \boxed{\frac{1}{16}\left[x - \frac{\sin 4x}{4} + \frac{\sin^3 2x}{3}\right] + C}$$

Eliminating Square Roots

4) $\displaystyle\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx$

Sol: Recall that $2\cos^2\theta = 1+\cos 2\theta$, so we have that $\sqrt{1+\cos 4x} = \sqrt{2\cos^2 2x} = \sqrt{2}\,|\cos 2x|$

so $\displaystyle\int_0^{\pi/4} \sqrt{1+\cos 4x}\, dx = \sqrt{2}\int_0^{\pi/4} |\cos 2x|\, dx$.

$\displaystyle = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx$ [since if $0<x<\frac{\pi}{4}$ then $0<2x<\frac{\pi}{2}$ and so $\cos 2x>0$]

$\displaystyle = \frac{\sqrt{2}}{2}\sin 2x\Big]_0^{\pi/4} = \boxed{\frac{\sqrt{2}}{2}}$

5) $\displaystyle\int \tan^4 x\, dx = \int \tan^2 x \cdot \tan^2 x\, dx = \int \tan^2 x(\sec^2 x - 1)\, dx$

$\displaystyle = \int \tan^2 x \cdot \sec^2 x\, dx - \int \tan^2 x\, dx$ $\qquad u = \tan x$

$\displaystyle = \int u^2 du - \int (\sec^2 x - 1)\, dx$ $\qquad du = \sec^2 x\, dx$

$\displaystyle = \frac{u^3}{3} - \tan x + x + C = \boxed{\frac{\tan^3 x}{3} - \tan x + x + C}$

ملحوظة: بنفس طريقة التكامل السابقة، يمكن إيجاد قانون اختزال (reduction formula) للتكامل $\int \tan^n x\, dx$ ومن ثم ايجاد التكامل السابق وغيره إذا $n \geq 2$ كالتالي :

$\displaystyle\int \tan^n x\, dx = \int \tan^{n-2} x \cdot \tan^2 x\, dx = \int \tan^{n-2} x \cdot (\sec^2 x - 1)\, dx$

$\displaystyle = \int \tan^{n-2} x\, \sec^2 x\, dx - \int \tan^{n-2} x\, dx = \boxed{\frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx}$

ويمكن تطبيق هذا القانون في المثال السابق كالتالي :

$$\int \tan^4 x\,dx = \frac{\tan^3 x}{3} - \int \tan^2 x\,dx = \frac{\tan^3 x}{3} - \left(\frac{\tan x}{1} - \int dx\right)$$

$$= \boxed{\frac{\tan^3 x}{3} - \tan x + x + C}$$

6) $\int \sec^3 x\,dx = \int \sec x \cdot \sec^2 x\,dx$

$u = \sec x \qquad dv = \sec^2 x\,dx$

$du = \sec x \tan x\,dx \qquad v = \tan x$

$$= \sec x \tan x - \int \tan^2 x \cdot \sec x\,dx = \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx$$

نفس التكامل الأصلي

$$= \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx$$

$$\Rightarrow 2\int \sec^3 x\,dx = \sec x \tan x + \ln|\sec x + \tan x| + C$$

$$\therefore \int \sec^3 x\,dx = \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right] + C$$

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx\,dx, \qquad \int \sin mx \cos nx\,dx, \qquad \text{and} \qquad \int \cos mx \cos nx\,dx$$

arise in many applications involving periodic functions. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the identities

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x], \tag{3}$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x], \tag{4}$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]. \tag{5}$$

These identities come from the angle sum formulas for the sine and cosine functions (Section 1.3). They give functions whose antiderivatives are easily found.

7) $\int \sin 3x \cos 5x \, dx = \frac{1}{2} \int [\sin(3-5)x + \sin(3+5)x] \, dx$

$= \frac{1}{2} \int [\sin(-2x) + \sin 8x] \, dx = \frac{1}{2} \int (\sin 8x - \sin 2x) \, dx$

$= \boxed{\frac{1}{2} \left(\frac{-\cos 8x}{8} + \frac{\cos 2x}{2} \right) + C}$

ملاحظة

Examples:

1) $\int_0^{\pi/6} \sqrt{1 + \sin x} \, dx \; * \; \frac{\sqrt{1 - \sin x}}{\sqrt{1 - \sin x}}$

$= \int_0^{\pi/6} \frac{\sqrt{1 - \sin^2 x}}{\sqrt{1 - \sin x}} \, dx = \int_0^{\pi/6} \frac{|\cos x|}{\sqrt{1 - \sin x}} \, dx = \int_0^{\pi/6} \frac{\cos x}{\sqrt{1 - \sin x}} \, dx$

$= -\int_1^{1/2} \frac{du}{\sqrt{u}} = -2\sqrt{u} \Big]_1^{1/2}$

$u = 1 - \sin x$

$du = -\cos x \, dx$

$x = 0 \rightarrow u = 1$

$x = \pi/6 \rightarrow u = \frac{1}{2}$

$= 2 \left[1 - \frac{1}{\sqrt{2}} \right] = \boxed{2 - \sqrt{2}}$

2) $\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1 - \cos x}} \, dx$

Recall that $\sin 2\theta = 2 \sin \theta \cos \theta$

$\Rightarrow \sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}$

and $2 \sin^2 \theta = 1 - \cos 2\theta$

$\therefore \; 2 \sin^2 \frac{x}{2} = 1 - \cos x$

عوضهم في التكامل

$= \int_{\pi/3}^{\pi/2} \frac{\left(2 \sin \frac{x}{2} \cos \frac{x}{2}\right)^2}{\sqrt{2 \sin^2 \frac{x}{2}}} \, dx$

$= \frac{4}{\sqrt{2}} \int_{\pi/3}^{\pi/2} \frac{\sin^2 \frac{x}{2} \cos^2 \frac{x}{2}}{\left|\sin \frac{x}{2}\right|} \, dx = 2\sqrt{2} \int_{\pi/3}^{\pi/2} \cos^2 \frac{x}{2} \sin \frac{x}{2} \, dx$

$$= 2\sqrt{2} * -2 \int_{\sqrt{3}/2}^{1/\sqrt{2}} u^2 \, du$$

take $u = \cos \frac{x}{2}$

$du = -\frac{1}{2} \sin \frac{x}{2} \, dx$

$-2 \, du = \sin \frac{x}{2} \, dx$

$x = \frac{\pi}{3} \longrightarrow u = \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{2} \longrightarrow u = \frac{1}{\sqrt{2}}$

$$= -4\sqrt{2} \left. \frac{u^3}{3} \right]_{\sqrt{3}/2}^{1/\sqrt{2}} =$$

$$= -\frac{4}{3}\sqrt{2} \left[\left(\frac{1}{\sqrt{2}}\right)^3 - \left(\frac{\sqrt{3}}{2}\right)^3 \right] = \boxed{\sqrt{\frac{3}{2}} - \frac{2}{3}}$$

3) $\int \sin^4 x \, dx = \int \left(\sin^2 x\right)^2 dx = \int \left(\frac{1 - \cos 2x}{2}\right)^2 dx$

$$= \frac{1}{4} \int \left(1 - 2\cos 2x + \cos^2 2x\right) dx$$

$$= \frac{1}{4} \left[x - \frac{2 \sin 2x}{2} + \frac{1}{2} \int 1 + \cos 4x \, dx \right]$$

$$= \boxed{\frac{x}{4} - \frac{\sin 2x}{4} + \frac{1}{8}\left(x + \frac{\sin 4x}{4}\right) + C}$$

4) $\int_{-\pi/2}^{\pi/2} \cos x \cos 7x \, dx = 2 \int_0^{\pi/2} \frac{1}{2}\left[\cos(1-7)x + \cos(1+7)x\right] dx$

[$\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$ التكامل للدالة الزوجية و زوجية دالة هي $\cos x \cos 7x$]

$$= \int_0^{\pi/2} \left(\cos 6x + \cos 8x\right) dx = \left[\frac{\sin 6x}{6} + \frac{\sin 8x}{8} \right]_0^{\pi/2} = \boxed{0}$$

5) $\int \sec^4 x \tan^2 x \, dx = \int \sec^2 x \tan^2 x \cdot \sec^2 x \, dx$

$u = \tan x$

$du = \sec^2 x \, dx$

$$= \int (1 + \tan^2 x) \tan^2 x \cdot \sec^2 x \, dx = \int (1 + u^2) u^2 \, du$$

$$= \frac{u^3}{3} + \frac{u^5}{5} + C = \boxed{\frac{\tan^3 x}{3} + \frac{\tan^5 x}{5} + C}$$