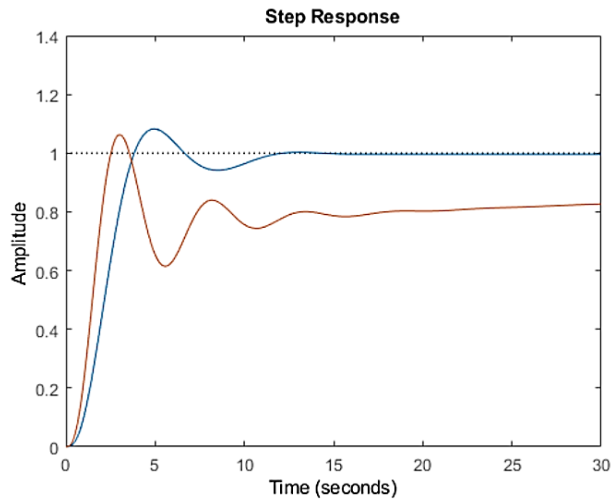
Design of a Servo System (Tracking System):

The tracking system is a control action aims to force the output response y (t) to follow the desired input r(t) with a required performance.



There are two cases for design the tracking system:

- Find the eigenvalues for the open loop system $|sI - A| = 0$
- Check if there is any eigenvalues at the origin or not i.e. find the Type number?
- Remember this:

TABLE 7.2 Relationships between input, system type, static error constants, and steady-state errors

Input	Steady-state error formula	Type 0 Static error constant	Type 0 Error	Type 1 Static error constant	Type 1 Error	Type 2 Static error constant	Type 2 Error
Step, $u(t)$	$\frac{1}{1+K_p}$	K_p = Constant	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	K_v = Constant	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	K_a = Constant	$\frac{1}{K_a}$

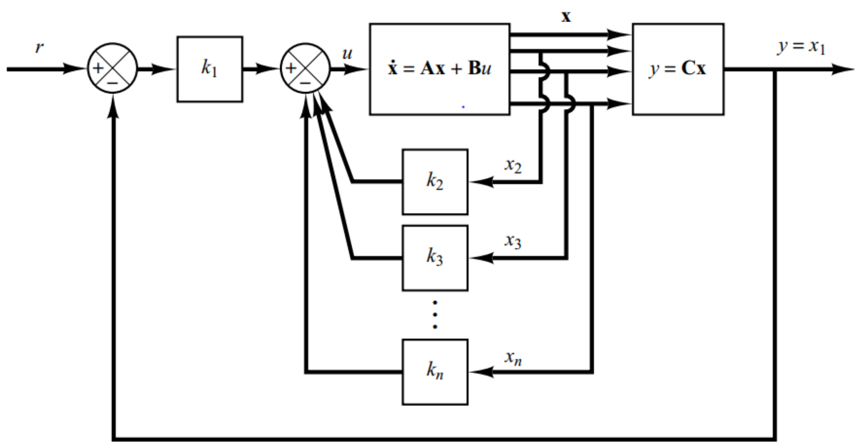
- If the steady state error is equal zero based on the Type number use the first case. Otherwise use the second case.

Case 1 : Design of Type 1 Servo System when the Plant Has an Integrator:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \qquad (1)$$

$$y = \mathbf{C}\mathbf{x} \qquad (2)$$

Here we assumed that y = x_1



the reference input *r* is a step function. In this system we use the following state-feedback control scheme:

$$u = -\begin{bmatrix} 0 & k_2 & k_3 & \cdots & k_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} + k_1(r - x_1)$$

$$= -\mathbf{K}\mathbf{x} + k_1 r \qquad (3)$$

where

$$\mathbf{K} = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}$$

Then, for t > 0, the system dynamics can be described by Equations (1) and (3), or

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x} + \mathbf{B}k_1 r \qquad (4)$$

The designed system will be an asymptotically stable system, $y(\infty)$ will approach the constant value r, and $u(\infty)$ will approach zero. Where r is a step input.

Notice that at steady state we have

$$\dot{\mathbf{x}}(\infty) = (\mathbf{A} - \mathbf{BK})\mathbf{x}(\infty) + \mathbf{B}k_1 r(\infty) \qquad 5$$

Noting that r(t) is a step input, we have $r(\infty) = r(t) = r(constant)$ for t>0. By subtracting Equation (5) from Equation (4), we obtain

$$\dot{\mathbf{x}}(t) - \dot{\mathbf{x}}(\infty) = (\mathbf{A} - \mathbf{BK})[\mathbf{x}(t) - \mathbf{x}(\infty)] \qquad 6$$

Define

$$\mathbf{x}(t) - \mathbf{x}(\infty) = \mathbf{e}(t)$$

Then Equation 6 becomes

$$\dot{\mathbf{e}} = (\mathbf{A} - \mathbf{BK})\mathbf{e} \qquad 7$$

- The design of the type 1 servo system here is converted to the design of an asymptotically stable regulator system such that **e**(t) approaches zero, given any initial condition **e**(0).
- If the system defined by Equation (1) is completely state controllable, then, by specifying the desired eigenvalues $\mu_1, \mu_2, \dots, \mu_n$ for the matrix $A - BK$, matrix K can be determined by the pole-placement technique or Linear Quadratic Regulator (LQR).

EXAMPLE 10–4 Design a type 1 servo system when the plant transfer function has an integrator. Assume that the plant transfer function is given by

The required performance is $T_s = 1$ seconds and $\%OS = \%10$. Assume that the system configuration is the same as that shown in Figure below and the reference input r is a step function. Obtain the unit-step response of the designed system.

$$\dot{x} = \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 0]$$

$$\zeta = \sqrt{\frac{\left(\ln \frac{PO}{100}\right)^2}{\pi^2 + \left(\ln \frac{PO}{100}\right)^2}} = 0.6$$

$$T_s = \frac{4}{\zeta\omega_n} = 1 \rightarrow \qquad \omega_n = 6.67 \text{ rad/s}$$

Based on the desired eigenvalues are:

$$s_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -4 \pm 5.33j$$

$$u = -(k_2 x_2) + k_1(r - x_1) = -\mathbf{Kx} + k_1 r$$

$$\mathbf{K} = [k_1 \quad k_2]$$

1. Check the controllability:

$M = [B \quad AB]=\begin{bmatrix} 0 & 2 \\ 1 & 10 \end{bmatrix}$

$|M| = -2$ Thus, the system is fully state controllable.

2. Find the type number for the system:

$$|sI - A| = 0$$

$$|sI - A| = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix} = \begin{bmatrix} s-1 & -2 \\ -5 & s-10 \end{bmatrix}$$

$$|sI - A| = s^2 - 11\,s = s(s-11)$$

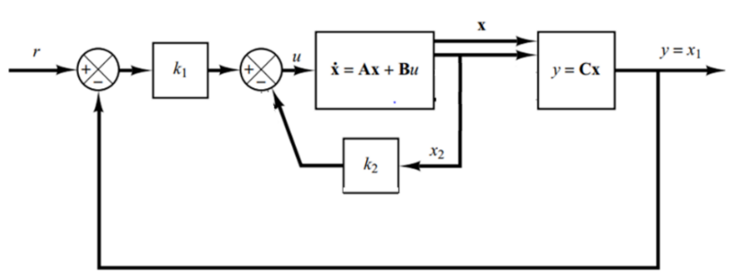
$$s_1 = 0, s_2 = 11$$

By the way the general form is:

$$s^2 + a_1\,s + a_2$$

$$a_1 = -11 \quad a_2 = 0$$

The type number is equal 1 and the desired input r is step input so we use the first case. Based on this use this control scheme:



3. Find the desired characteristic equation

$$(s+4+5.33j)(s+4+5.33j)=s^2+8\,s+44.4$$

By the way the general form is:

$$s^2+\alpha_1\,s+\alpha_2$$

$$\alpha_1=8 \qquad \alpha_2=44.4$$

4. Find the similarity matrix T

T=MW where W= $\begin{bmatrix} \alpha_1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -11 & 1 \\ 1 & 0 \end{bmatrix}$

$$T=\begin{bmatrix} 0 & 2 \\ 1 & 10 \end{bmatrix}\begin{bmatrix} -11 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$

$$T^{-1}=\begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}$$

5. Find the gain matrix:

$$K=[(\alpha_2-a_2) \quad (\alpha_1-a_1)]T^{-1}=[(44.4-0) \;\; (8--11)]\begin{bmatrix} 0.5 & 0 \\ 0.5 & 1 \end{bmatrix}$$

$$K=[31.7 \quad 19]$$

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EXAMPLE 10–4 Design a type 1 servo system when the plant transfer function has an integrator. Assume that the plant transfer function is given by

The desired closed-loop poles are $s=-2\pm 2\sqrt{3}j$ and $s=-10$. Assume that the system configuration is the same as that shown in Figure below and the reference input r is a step function. Obtain the unit-step response of the designed system.

Then the state-space representation of the system becomes

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$$

$$y = \mathbf{C}\mathbf{x}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{C} = [1 \quad 0 \quad 0]$$

$$u = -(k_2x_2 + k_3x_3) + k_1(r - x_1) = -\mathbf{K}\mathbf{x} + k_1 r$$

where

$$\mathbf{K} = [k_1 \quad k_2 \quad k_3]$$

6. Check the controllability:

$$M = [B \quad AB \quad A^2B] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & -3 & 7 \end{bmatrix}$$

$|M| = -1$ Thus, the system is fully state controllable.

7. Find the type number for the system:

$$|sI - A| = 0$$

$$|sI - A| = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 0 & 2 & s+3 \end{bmatrix}$$

$$|sI - A| = s^3 + 3\,s^2 + 2\,s = s(s^2 + 3s + 2) = s\,(s+1)(s+2)$$

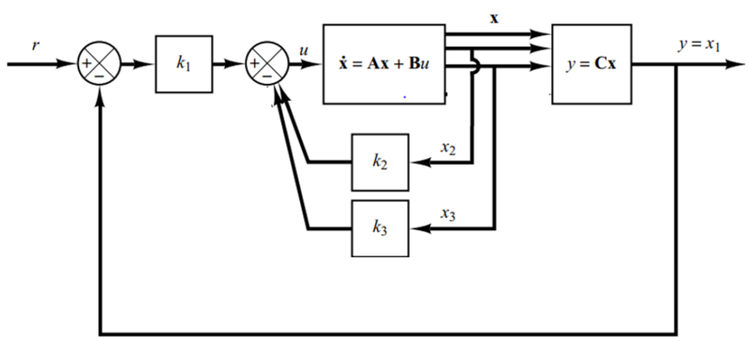
$$s_1 = 0 \quad , s_2 = -1, \;\; s_3 = -2$$

By the way the general form is:

$$s^3 + a_1\,s^2 + a_2\,s + a_3$$

$$a_1 = 3 \quad a_2 = 2 \quad a_3 = 0$$

The type number is equal 1 and the desired input r is step input so we use the first case. Based on this use this control scheme:



8. Find the desired characteristic equation

$$(s + 2 + 2\sqrt{3}j)(s + 2 - 2\sqrt{3}j)\ (s + 10) = s^3 + 14\,s^2 + 56\,s + 160$$

By the way the general form is:

$$s^3 + \alpha_1\,s^2 + \alpha_2\,s + \alpha_3$$

$$\alpha_1 = 14 \quad \alpha_2 = 56 \quad \alpha_3 = 160$$

9. Find the gain matrix:

$$K = [(\alpha_3 - a_3) \quad (\alpha_2 - a_2) \quad (\alpha_1 - a_1)] = [(160 - 0) \quad (56 - 2) \quad (14 - 3)]$$

$$K = [160 \quad 54 \quad 11]$$