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MATH1321

Calculus 2

Chapter 11.1

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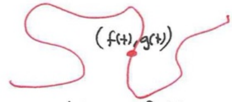
بسبب ضيق الوقت ما لخصت
التشابتر، هاد تلخيص دكتور
عبدالرحيم موسى وضفت عليه
النوتس الي بحكيهم خلال الشرح
ورح يكون هيك لآخر المادة

GOOD LUCKKKKK

11.1 Parametrization of Plane Curves

* We may describe the movement of a particle in the xy plane at position t by $(x(t), y(t)) = (f(t), g(t))$



position of the particle at time t
not a function

Def If x and y are given as functions

$$x = f(t) \text{ and } y = g(t), \quad t \in I,$$

Then the set of points $(x, y) = (f(t), g(t))$ is a parameteric curve.

Note that [1] $x = f(t)$ and $y = g(t)$ are called parametric equations.

[2] the variable t is called the parameter of the curve.

[3] the interval I is called the parameter interval.

$\Rightarrow$ If $I = [a, b]$ closed interval, then
the point $(f(a), g(a))$ is the initial point and
the point $(f(b), g(b))$ is the terminal point.

[4] We say that we have parametrized the curve, if we find [1] and [3]. That is [1] and [3] give a parametrization of the curve.

Exp Given the parametric equation and parameter interval:

$$x = t^2, \quad y = t + 1, \quad -\infty < t < \infty$$

[1] Find the Cartesian ($\rightarrow$ algebraic) equation by eliminating the parameter t

[2] Idenitify the particle's path by sketching the cartesian equation

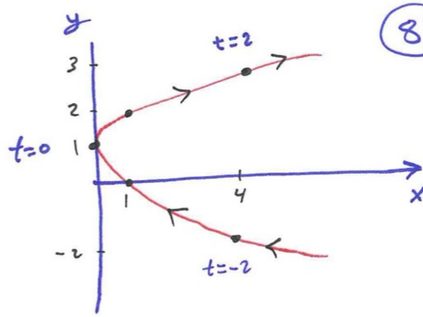
[3] Find the direction of motion

[1] Cartesian equation: $x = t^2 = (y-1)^2 \Leftrightarrow x = (y-1)^2$

Note that sometimes it's difficult or even impossible to eliminate the parameter t.

The curve that represents the particle movement.

t	-3	-2	-1	0	1	2	3
x	9	4	1	0	1	4	9
y	-2	-1	0	1	2	3	4

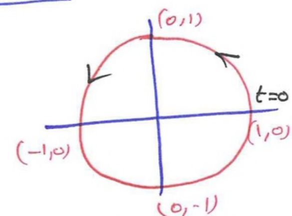


Exp Graph the parametric curve of $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$

- We can eliminate the parameter t by:

$$x^2 + y^2 = \cos^2 t + \sin^2 t = 1 \iff \boxed{x^2 + y^2 = 1} \text{ cartesian equation}$$

- Initial point is $(\cos 0, \sin 0) = (1, 0)$
- Terminal point is $(\cos 2\pi, \sin 2\pi) = (1, 0)$
- $t = \pi \Rightarrow$ the position is $(-1, 0)$



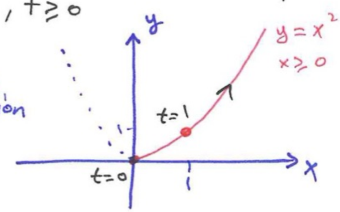
Direction: counterclockwise

Exp Graph the particle's movement and direction if its parametric equation and parameter interval is [1] $x = \sqrt{t}$, $y = t$, $t \ge 0$

- We can eliminate the parameter t

$$y = t = x^2 \iff \boxed{y = x^2} \text{ Cartesian equation}$$

- Initial point is $(\sqrt{0}, 0) = (0, 0)$
- No terminal point
- $t = 1 \Rightarrow$ the position is $(1, 1)$

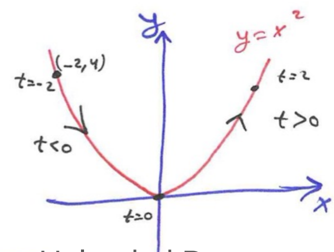


[2] $x = t$, $y = t^2$, $-\infty < t < \infty$

- We can eliminate the parameter t

$$y = t^2 = x^2 \iff \boxed{y = x^2} \to \text{Cartesian equation}$$

- no initial point
- no terminal point



Exp Find a parametrization for the line passes throw the points (a,b) and (c,d).

- A Cartesian equation is $y-b=m(x-a)$ where the slope $m=\frac{d-b}{c-a}$, $c\neq a$
- Set the parameter $t=x-a$
- Hence, $x=a+t$, $y=b+mt$, $-\infty<t<\infty$

parameterizes the line.

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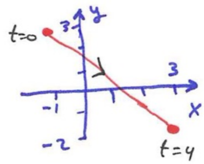
the line segment with endpoints $(-1,3)$ and $(3,-2)$ $m=\frac{-2-3}{3+1}=\frac{-5}{4}$

$\{x=-1+t,\ y=3-\frac{5}{4}t,\ 0\le t\le 4\}$

$\{x=-1+4t,\ y=3-5t,\ 0\le t\le 1\}$



Both parametrization give the same segment

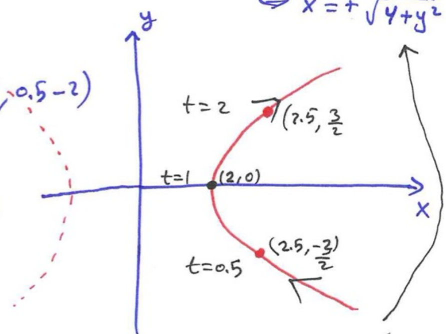
Exp sketch and identify the path by the point $P(x,y)$ if

$$x=t+\frac{1}{t},\quad y=t-\frac{1}{t},\quad t>0$$

- We can eliminate the parameter t by:

$$\left.\begin{aligned}x+y&=2t\\ x-y&=\frac{2}{t}\end{aligned}\right\}\rightarrow (x+y)(x-y)=4 \iff x^2-y^2=4 \iff x^2=4+y^2 \iff x=+\sqrt{4+y^2}$$

- at $t=0.5 \Rightarrow$ the position is $(0.5+2, 0.5-2)$
  $\Rightarrow (2.5, \frac{-3}{2})$
- at $t=2 \Rightarrow$ the position is $(2.5, \frac{3}{2})$



$x>0$ since $t>0$

Note that

$\{x=t+\frac{1}{t},\ y=t-\frac{1}{t},\ t>0\}$

$\{x=\sqrt{4+t^2},\ y=t,\ -\infty<t<\infty\}$

$\{x=2\sec t,\ y=2\tan t,\ \frac{-\pi}{2}<t<\frac{\pi}{2}\}$

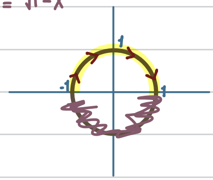
are all different parametrization for the same curve.

Ex. $x = t, \quad y = \sqrt{1-t^2}, \quad -1 \leq t \leq 0$?

A) Cartesian eq. $y = \sqrt{1-t^2}$ so $y = \sqrt{1-x^2}$

B) $y^2 = 1 - x^2$ so $y^2 + x^2 = 1$

IP $-1 \leq t \leq 0$ TP



IP: $t = -1 \quad (x, y) = (-1, 0)$

TP: $t = 0 \quad (x, y) = (0, 1)$

Ex. $x = -\sec t, \quad y = \tan t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2}$?

A) using $\sec^2 t - \tan^2 t = 1$

$x^2 - y^2 = (-\sec t)^2 - (\tan t)^2 = 1$

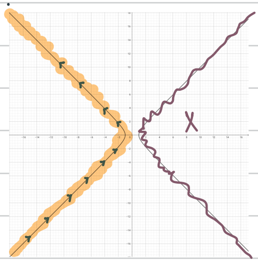
$= \sec^2 t - \tan^2 t = 1$

so the cartesian eq. is $x^2 - y^2 = 1$.

$x = -\sec t < 0$

sin | All +sec +cos

tan | cos +cos +sec



Ex. $x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0$?

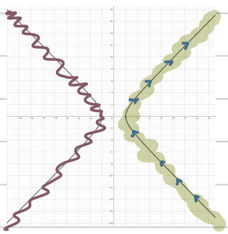
$x = t + \frac{1}{t}$ $\qquad$ $x = t + \frac{1}{t}$

$y = t - \frac{1}{t}$ $+$ $\qquad$ $y = t - \frac{1}{t}$ $-$

$x + y = 2t$ $\qquad$ $x - y = \frac{2}{t}$

$(x+y)(x-y) = 4$ so $x^2 - y^2 = 4$

when $t > 0 \longrightarrow x > 0$



C) direction: $t_0 = \frac{1}{2} \longrightarrow x = 2.5$ and $y = -1.5$

at t_0 $(x, y) = (2.5, -1.5)$

$t_1 = 2 \longrightarrow x = 2$ and $y = 1.5$

at t_1 $(x, y) = (2, 1.5)$

Remark: we can write more than one Parametrization for the same curve.

$\left\{ \begin{array}{l} x = f(t) \\ y = g(t) \\ t \in I \end{array} \right\}$ $\rightarrow$ Parametrization eq.

Par. interval $\rightarrow$ $t \in I$

Ex. Parametrization 1: $\{ x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0 \}$

Parametrization 2: $\{ x = \sqrt{4 + t^2}, \quad y = t, \quad -\infty < t < \infty \}$

Parametrization 3: $\{ x = 2\sec t, \quad y = 2\tan t, \quad -\frac{\pi}{2} < t < \frac{\pi}{2} \}$

