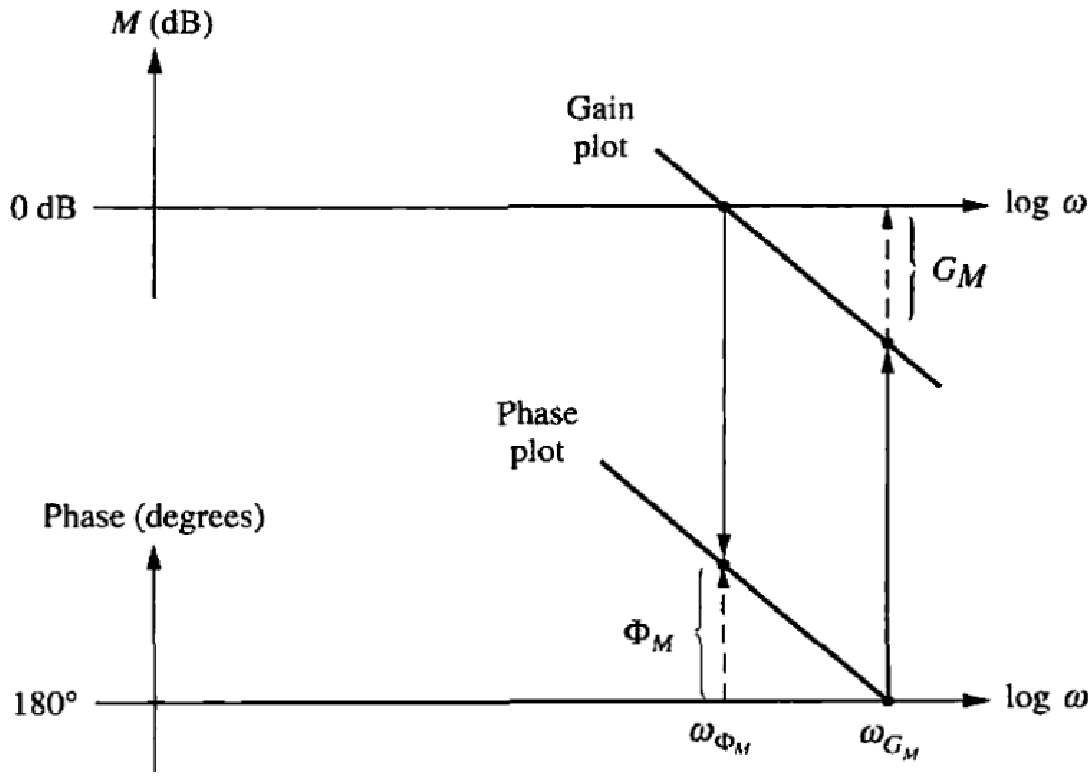
Analysis of Bode Plots

Informal definitions:

- The **gain margin** is the factor by which the gain can be increased before instability results.
- The **phase margin** is the amount of phase by which $G(j\omega)$ exceeds -180 degrees when $|KG(j\omega)|=1$

- These are easily measured on Bode diagrams.



Gain Margin

- The greater the **Gain Margin** (GM), the greater the stability of the system. The gain margin refers to the amount of gain, which can be increased or decreased without making the system unstable. It is usually expressed as a magnitude in dB.
- The gain margin is calculated directly from the Bode plot (as shown in the diagram below). This is done by calculating the vertical distance between the magnitude curve (on the Bode magnitude plot) and the x-axis at the frequency where the Bode phase plot = -180° or 180°. This point is known as the **phase crossover frequency**.

Gain Margin Formula

- The **formula for Gain Margin** (GM) can be expressed as:

$$GM = 0 - G\ dB$$

- Where *G* is the gain. This is the magnitude (in dB) as read from the vertical axis of the magnitude plot at the phase crossover frequency. In our example shown in the graph above, the Gain (*G*) is 20. Hence using our formula for gain margin, the gain margin is equal to 0 – 20 dB = -20 dB (unstable).

Phase Margin

- The greater the **Phase Margin** (PM), the greater will be the stability of the system. The phase margin refers to the amount of phase, which can be increased or decreased without making the system unstable. It is usually expressed as a phase in degrees.
- The phase margin is calculated directly from the Bode plot (as shown in the diagram above). This is done by calculating the vertical distance between the phase curve (on the Bode phase plot) and the x-axis at the frequency where the Bode magnitude plot = 0 dB. This point is known as the **gain crossover frequency**.

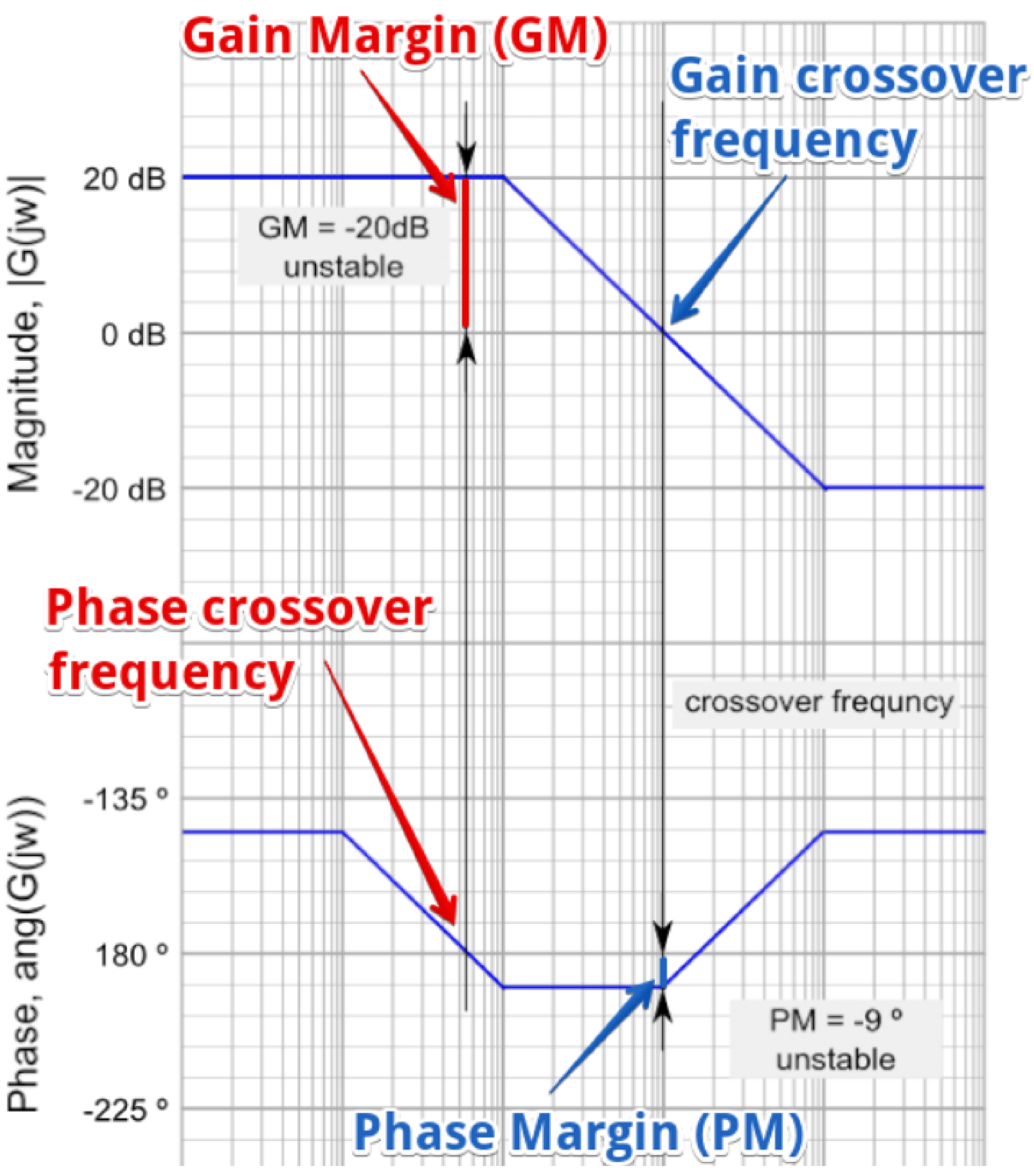
Phase Margin Formula

- The **formula for Phase Margin** (PM) can be expressed as:

$$PM = \phi - (-180^{\circ})$$

- Where ϕ is the phase lag (a number less than 0). This is the phase as read from the vertical axis of the phase plot at the gain crossover frequency.

- In our example shown in the graph above, the phase lag is -189°. Hence using our formula for phase margin, the phase margin is equal to -189° – (-180°) = -9° (unstable).



Steady-State Error Characteristics from Frequency Response

TABLE 7.2 Relationships between input, system type, static error constants, and steady-state errors

Input	Steady-state error formula	Type 0 Static error constant	Type 0 Error	Type 1 Static error constant	Type 1 Error	Type 2 Static error constant	Type 2 Error
Step, $u(t)$	$\frac{1}{1+K_p}$	$K_p = \text{Constant}$	$\frac{1}{1+K_p}$	$K_p = \infty$	0	$K_p = \infty$	0
Ramp, $tu(t)$	$\frac{1}{K_v}$	$K_v = 0$	∞	$K_v = \text{Constant}$	$\frac{1}{K_v}$	$K_v = \infty$	0
Parabola, $\frac{1}{2}t^2u(t)$	$\frac{1}{K_a}$	$K_a = 0$	∞	$K_a = 0$	∞	$K_a = \text{Constant}$	$\frac{1}{K_a}$

To find K_p, consider the following Type 0 system:

$$G(s) = K\frac{\prod_{i=1}^{n}(s+z_i)}{\prod_{i=1}^{m}(s+p_i)} \tag{10.74}$$

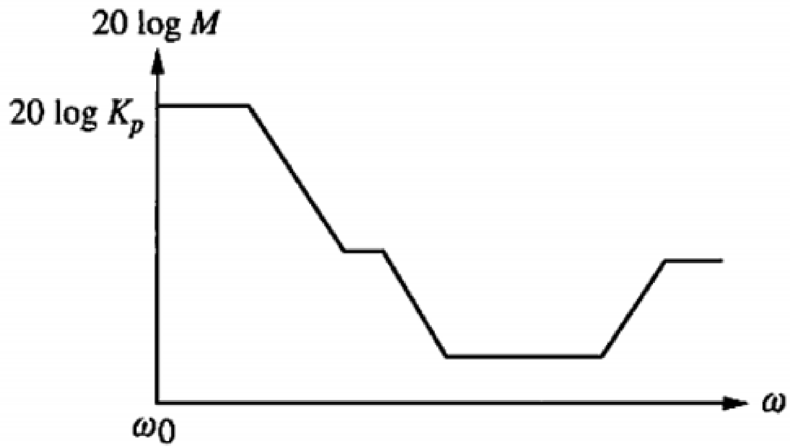
A typical unnormalized and unscaled Bode log-magnitude plot is shown in Figure 10.51(a). The initial value is

$$20\log M = 20\log K\frac{\prod_{i=1}^{n} z_i}{\prod_{i=1}^{m} p_i} \tag{10.75}$$

But for this system

$$K_p = K\frac{\prod_{i=1}^{n} z_i}{\prod_{i=1}^{m} p_i} \tag{10.76}$$

which is the same as the value of the low-frequency axis. Thus, for an unnormalized and unscaled Bode log-magnitude plot, the low-frequency magnitude is $20\log K_p$ for a Type 0 system.



(a)

Velocity Constant

To find K_v for a Type 1 system, consider the following open-loop transfer function of a Type 1 system:

$$G(s) = K \frac{\prod_{i=1}^{n}(s+z_i)}{s\prod_{i=1}^{m}(s+p_i)} \tag{10.77}$$

A typical unnormalized and unscaled Bode log-magnitude diagram is shown in Figure 10.51(b) for this Type 1 system. The Bode plot starts at

$$20\log M = 20\log K \frac{\prod_{i=1}^{n} z_i}{\omega_0 \prod_{i=1}^{m} p_i} \tag{10.78}$$

The initial −20 dB/decade slope can be thought of as originating from a function,

$$G'(s) = K \frac{\prod_{i=1}^{n} z_i}{s\prod_{i=1}^{m} p_i} \tag{10.79}$$

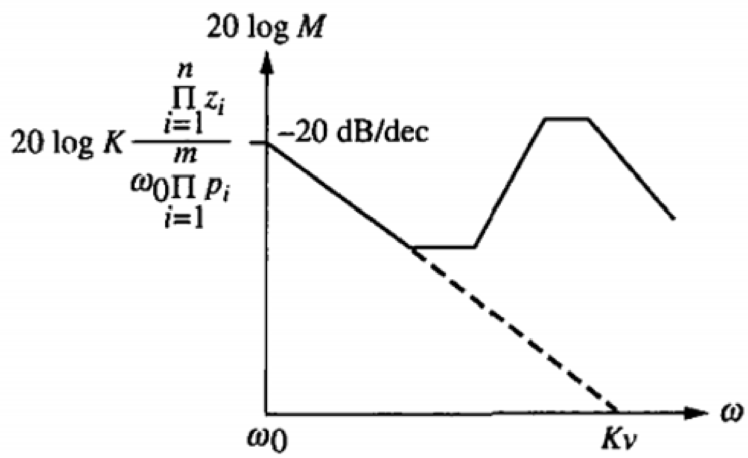
$G'(s)$ intersects the frequency axis when

$$\omega = K\frac{\prod_{i=1}^{n} z_i}{\prod_{i=1}^{m} p_i} \tag{10.80}$$

But for the original system (Eq. (10.77)),

$$K_v = K\frac{\prod_{i=1}^{n} z_i}{\prod_{i=1}^{m} p_i} \tag{10.81}$$

which is the same as the frequency-axis intercept, Eq. (10.80). Thus, we can find K_v by extending the initial -20 dB/decade slope to the frequency axis on an unnormalized and unscaled Bode diagram. The intersection with the frequency axis is K_v.



(*b*)

Acceleration Constant

To find K_a for a Type 2 system, consider the following:

$$G(s) = K\frac{\prod_{i=1}^{n}(s+z_i)}{s^2\prod_{i=1}^{m}(s+p_i)} \tag{10.82}$$

A typical unnormalized and unscaled Bode plot for a Type 2 system is shown in Figure 10.51(c). The Bode plot starts at

$$20\log M = 20\log K\frac{\prod_{i=1}^{n}z_i}{\omega_0^2\prod_{i=1}^{m}p_i} \tag{10.83}$$

The initial −40 dB/decade slope can be thought of as coming from a function,

$$G'(s) = K\frac{\prod_{i=1}^{n}z_i}{s^2\prod_{i=1}^{m}p_i} \tag{10.84}$$

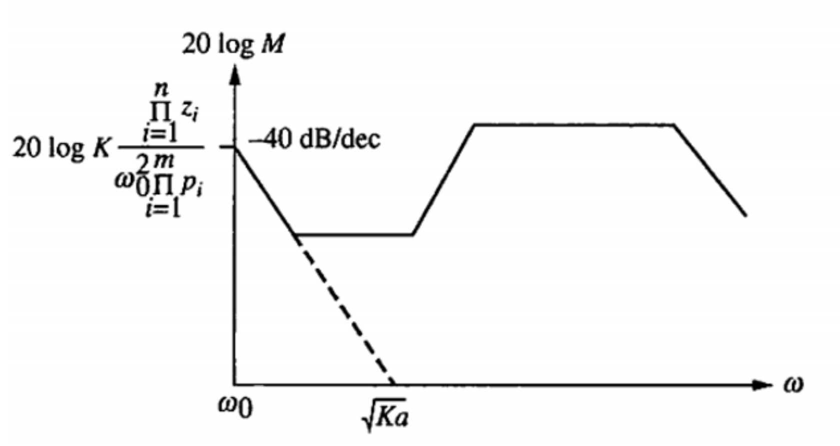
$G'(s)$ intersects the frequency axis when

$$\omega = \sqrt{K\frac{\prod_{i=1}^{n}z_i}{\prod_{i=1}^{m}p_i}} \tag{10.85}$$

But for the original system (Eq. (10.82)),

$$K_a = K\frac{\prod_{i=1}^{n}z_i}{\prod_{i=1}^{m}p_i} \tag{10.86}$$

Thus, the initial −40 dB/decade slope intersects the frequency axis at $\sqrt{K_a}$.



(c)

Relation Between Closed-Loop Transient and Closed-Loop Frequency Responses

Consider the following second order system

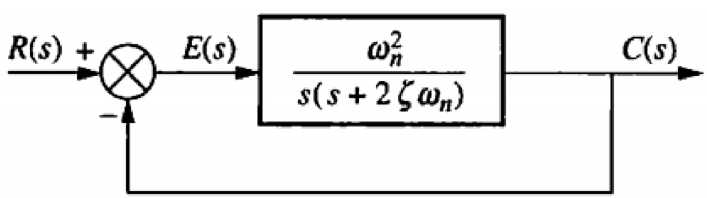


FIGURE 10.38 Second-order closed-loop system

$$\frac{C(s)}{R(s)} = T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{10.49}$$

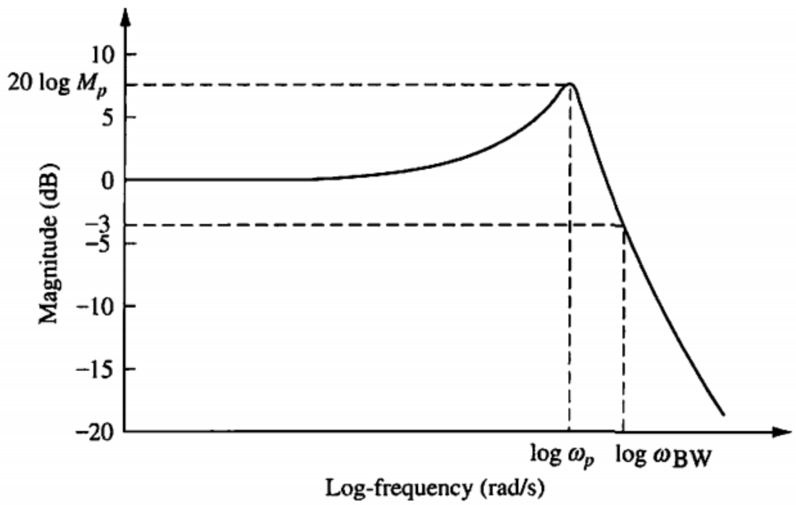
Let us now find the frequency response of Eq. (10.49), define characteristics of this response, and relate these characteristics to the transient response. Substituting $s = j\omega$ into Eq. (10.49), we evaluate the magnitude of the closed-loop frequency response as

$$M = |T(j\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}} \tag{10.51}$$

A representative sketch of the log plot of Eq. (10.51) is shown in Figure 10.39.

We now show that a relationship exists between the peak value of the closed-loop magnitude response and the damping ratio. Squaring Eq. (10.51), differentiating with respect to ω^2, and setting the derivative equal to zero yields the maximum value of M, M_p, where

$$\boxed{M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}} \tag{10.52}$$

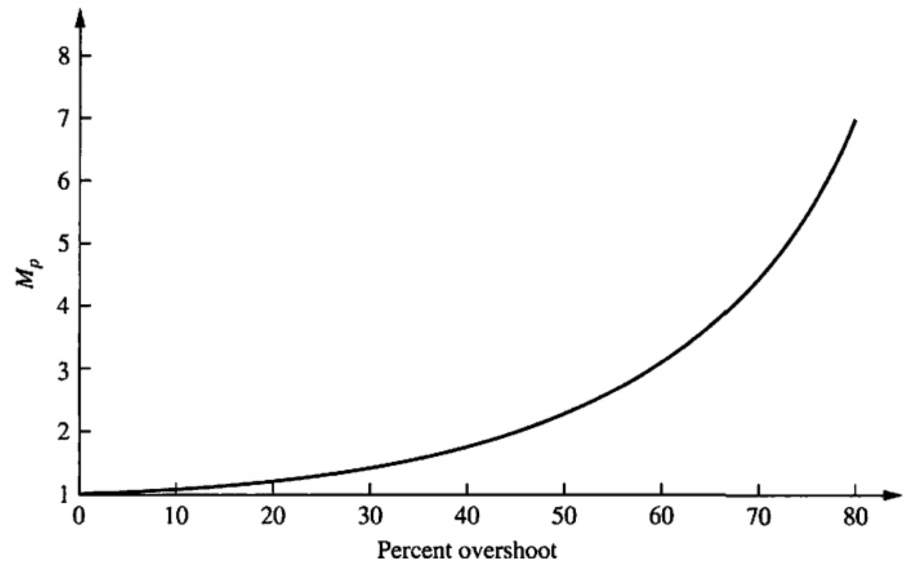


at a frequency, ω_p, of

$$\boxed{\omega_p = \omega_n\sqrt{1-2\zeta^2}} \tag{10.53}$$

Response Speed and Closed-Loop Frequency Response

Another relationship between the frequency response and time response is between the speed of the time response (as measured by settling time, peak time, and rise time) and the *bandwidth* of the closed-loop frequency response, which is defined here as the frequency, ω_{BW}, at which the magnitude response curve is 3 dB down from its value at zero frequency (see Figure 10.39).



The bandwidth of a two-pole system can be found by finding that frequency for which $M = 1/\sqrt{2}$ (that is, −3 dB) in Eq.(10.51). The derivation is left as an exercise for the student. The result is

$$\omega_{BW} = \omega_n\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}} \tag{10.54}$$

To relate ω_{BW} to settling time, we substitute $\omega_n = 4/T_s\zeta$ into Eq. (10.54) and obtain

$$\omega_{BW} = \frac{4}{T_s\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}} \tag{10.55}$$

Similarly, since, $\omega_n = \pi/(T_p\sqrt{1-\zeta^2})$,

$$\omega_{BW} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}} \tag{10.56}$$

To relate the bandwidth to rise time, T_r, we use Figure 4.16, knowing the desired ζ and T_r. For example, assume $\zeta = 0.4$ and $T_r \approx 0.2$ second. Using Figure 4.16, the ordinate $T_r\omega_n = 1.463$, from which $\omega_n = 1.463/0.2 = 7.315$ rad/s. Using Eq. (10.54), $\omega_{BW} =$ 10.05 rad/s. Normalized plots of Eqs. (10.55) and (10.56) and the relationship between bandwidth normalized by rise time and damping ratio are shown in Figure 10.41.

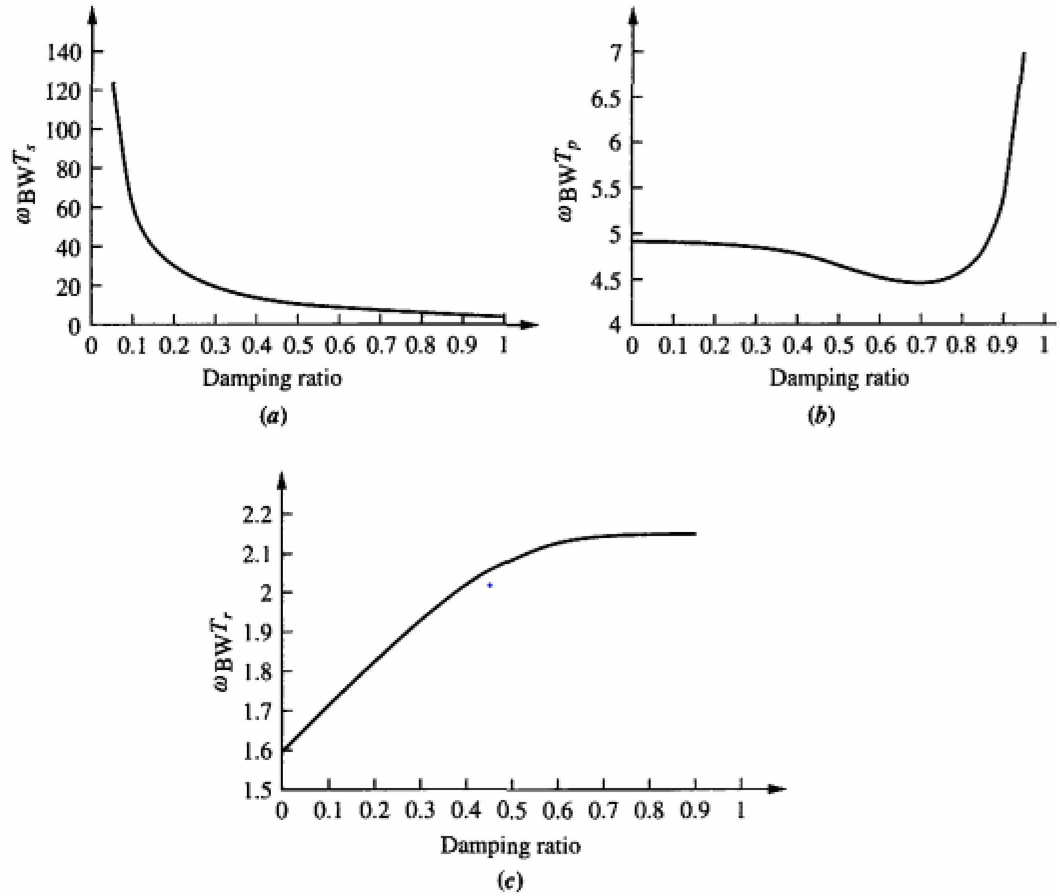


FIGURE 10.41 Normalized bandwidth vs. damping ratio for **a.** settling time; **b.** peak time; **c.** rise time

Relation Between Closed-Loop Transient and Open-Loop Frequency Responses

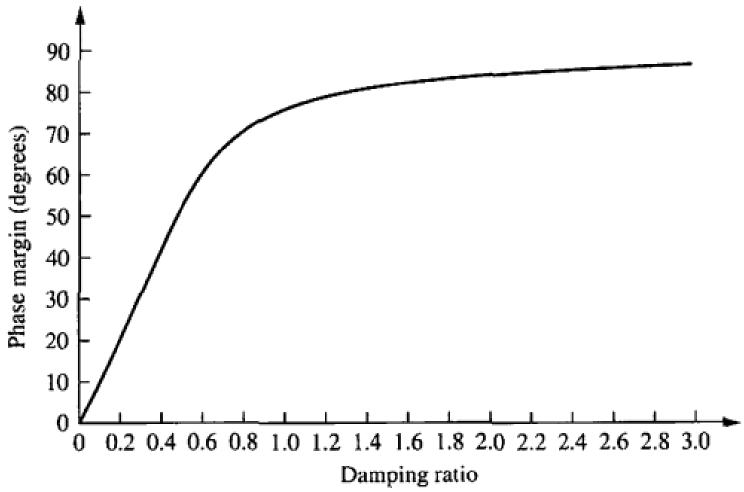
Damping Ratio from Phase Margin

Let us now derive the relationship between the phase margin and the damping ratio. This relationship will enable us to evaluate the percent overshoot from the phase margin found from the open-loop frequency response.

The difference between the angle of Eq. (10.72) and $-180°$ is the phase margin, ϕ_M. Thus,

$$\Phi_M = 90 - \tan^{-1}\frac{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}{2\zeta} = \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}} \quad (10.73)$$

Equation (10.73), plotted in Figure 10.48, shows the relationship between phase margin and damping ratio.



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