1.2 Row Echelon Form

* **Note**: we can not reduce $n \times n$ linear system to strict triangular form if, at any stage, all possible choices for a pivot element are 0 in a given column.

Exp Consider the system represented by the augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 1 & 1 & 2 & 2 & 4 & 1 \end{array}\right] \xrightarrow[R_5 - R_1]{\substack{R_1 + R_2 \\ 2R_1 + R_2}} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 3 & 0 \end{array}\right]$$

- All possible choices for a pivot in 2^{nd} column are 0, so it is not possible to reduce this system to strict triangular form.
- To simplify the system, we move to the 3^{rd} column

$$\xrightarrow[\substack{R_4 - R_2 \\ R_5 - R_2}]{R_3 - 2R_2} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_5 - R_3]{R_4 - R_3}$$

- All possible choices for a pivot in the 4^{th} column are 0, so we move to the 5^{th} column:

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{array}\right]$$

- So this is not strict triangular form but instead is echelon form.
- This system is inconsistent because of the last two equations.

Def A matrix is said to be in row echelon form if

1. the first nonzero entry in each nonzero row is 1
2. the K^{th} nonzero row has less number of leading zero entries than row $K+1$.
3. the zero rows "if exist" are below the nonzero rows.

Exp The following matrices are in row echelon form:

$$\begin{bmatrix} \boxed{1} & 2 & 3 & 0 \\ 0 & 0 & \boxed{1} & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \boxed{1} & 5 & 6 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \boxed{1} & 3 & 5 \\ 0 & \boxed{1} & 2 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

- x_2, x_4 are free variables
- Infinitely Many solution if the system is consistent

- x_2 is free v.
- Infinitely Many solution if the system is consistent

- $\exists$ unique solution if the system is consistent.

Exp The following matrices are not in row echelon form

$$\begin{bmatrix} \boxed{3} & 2 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & \boxed{2} \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

condition 1 fails

condition 2 fails

condition 3 fails

Def The process of using row operations I, II, III to transform a linear system to one whose augmented matrix is in row echelon form is called Gaussian elimination.

Notes 1. If the row echelon form of the augmented matrix contains a row of the form $[0 \ 0 \cdots 0 \mid 1]$, then the system is inconsistent

Exp Examples of inconsistent systems:

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 0 & 3 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 5 & 6 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

2. otherwise the system is consistent.

3. If the system is consistent and the nonzero rows of the row echelon form of the matrix form a strictly triangular system, then the system has a unique solution.

Exp
$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right], \quad \left[\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

Exp Use Gaussian elimination to solve the system 1
$$\begin{aligned} x_1 - 2x_2 &= 3 \\ 2x_1 - x_2 &= 9 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 2 & -1 & 9 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 3 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 1 & 1 \end{array}\right]$$

The system is consistent and has no free variables, so the system has unique solution "since it is in strict triangular form. So we use back substitution:

$x_2 = 1$ and $x_1 - 2 = 3 \Rightarrow x_1 = 5 \Rightarrow$ The solution is $(5,1)$

2
$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 1 \\ -x_1 - x_2 \qquad\qquad + x_5 &= -1 \\ -2x_1 - 2x_2 \qquad\qquad + 3x_5 &= 1 \\ x_3 + x_4 + 3x_5 &= 3 \\ x_1 + x_2 + 2x_3 + 2x_4 + 4x_5 &= 4 \end{aligned}$$

$$\Rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & 0 & 0 & 3 & 1 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 1 & 1 & 2 & 2 & 4 & 4 \end{array}\right] \xrightarrow[\substack{2R_1 + R_2 \\ R_5 - R_1}]{R_1 + R_2}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 2 & 2 & 5 & 3 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 0 & 0 & 1 & 1 & 3 & 3 \end{array}\right] \xrightarrow[\substack{R_4 - R_2 \\ R_5 - R_2}]{R_3 - 2R_2} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[R_5 - R_3]{R_4 - R_3}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent and has two free variables x_2 and x_4. The system is then has infinitely many solution.

Let $x_2 = \alpha$ and $x_4 = \beta$. $\Rightarrow x_5 = 3$

$x_3 + \beta + 6 = 0 \Rightarrow x_3 = -6 - \beta$

$x_1 + \alpha + (-6 - \cancel{\beta}) + \cancel{\beta} + 3 = 1 \Rightarrow x_1 = 4 - \alpha$

- The set of solutions is $\{(4-\alpha, \alpha, -6-\beta, \beta, 3) : \alpha, \beta \in \mathbb{R}\}$

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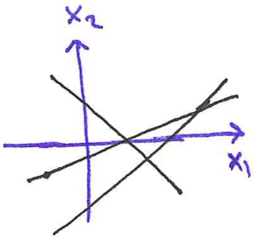
* **Overdetermined Systems:** A linear system is overdetermined if $m > n$ "equations more than unknowns"

Exp $\boxed{a}$

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - x_2 &= 3 \\ -x_1 + 2x_2 &= -2 \end{aligned} \Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 3 \\ -1 & 2 & -2 \end{array}\right] \xrightarrow[R_3 + R_1]{R_2 - R_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -2 & 2 \\ 0 & 3 & -1 \end{array}\right]$$



$$\Rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 3 & -1 \end{array}\right] \xrightarrow{R_3 - 3R_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{array}\right]$$

The system is inconsistent.

$\boxed{b}$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ 4x_1 + 3x_2 + 3x_3 &= 4 \\ 2x_1 - x_2 + 3x_3 &= 5 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 2 & -1 & 3 & 5 \end{array}\right] \xrightarrow[R_4 - 2R_1]{R_2 - 2R_1,\ R_3 - 4R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & -5 & 1 & 3 \end{array}\right]$$

$$\xrightarrow[R_4 - R_2]{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & -5 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 \end{array}\right] \overset{R_3}{\underset{R_4}{\Longleftrightarrow}} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 1 & \frac{3}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent and has a unique solution since the nonzero rows form a strict triangular form. $\left(\frac{1}{10}, \frac{-3}{10}, \frac{3}{2}\right)$

$\boxed{c}$

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ 4x_1 + 3x_2 + 3x_3 &= 4 \\ 3x_1 + x_2 + 2x_3 &= 3 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & -1 & 1 & 2 \\ 4 & 3 & 3 & 4 \\ 3 & 1 & 2 & 3 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{1}{5} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The system is consistent with x_3 free variable. so it has infinitely many solution. That is the solution set is $\left\{\left(1 - \frac{6\alpha}{10}, \frac{-2\alpha}{10}, \alpha\right) : \alpha \in \mathbb{R}\right\}$

Underdetermined Systems: A linear system is undetermined if $m<n$ " equations less than unknowns"

Exp Solve the following underdetermined systems:

[a] $x_1 + 2x_2 + x_3 = 1$
$2x_1 + 4x_2 + 2x_3 = 3$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 4 & 2 & 3 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

The system is inconsistent.

[b] $x_1 + x_2 + x_3 + x_4 + x_5 = 2$
$x_1 + x_2 + x_3 + 2x_4 + 2x_5 = 3$
$x_1 + x_2 + x_3 + 2x_4 + 3x_5 = 2$

$$\Rightarrow \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right] \begin{array}{l} R_2-R_1 \\ R_3-R_1 \end{array}$$

$$\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right] \xrightarrow{R_3-R_2} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

The system is consistent and has two free variables x_2 and x_3. So it has infinitely many solutions.

So we continuou the elimination to arrive the reduced row echelon form

$$\xrightarrow[R_2-R_3]{R_1-R_3} \left[\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{R_1-R_2} \underbrace{\left[\begin{array}{ccccc|c} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right]}_{\text{reduced row echelon form}}$$

Let $x_2 = \alpha$, $x_3 = \beta$.

$\Rightarrow x_5 = -1 \quad \Rightarrow x_4 = 2 \quad \Rightarrow x_1 = 1 - \alpha - \beta$

The set of solutions is $\{(1-\alpha-\beta, \alpha, \beta, 2, -1) : \alpha, \beta \in \mathbb{R}\}$

Note that [1] underdetermined systems can be inconsistent

[2] " " " " consistent with infinitely many solution.

[3] It is not possible for underdetermined system to have unique solution.

[4] In case where the row echelon form of a consistent system has free variables, we tranform the resulting matrix to the reduced row echelon form, and this process is called Gauss-Jordan reduction.

Def A matrix is said to be reduced row echelon form if

[1] The matrix is in row echelon form.

[2] The first nonzero entry in each row is the only nonzero entry in its column.

Exp Examples of reduced row echelon form:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Exp Use Gauss-Jordan reduction to solve the system

$$\begin{aligned} x_1 + 2x_2 - 3x_3 + x_4 &= 1 \\ -x_1 - x_2 + 4x_3 - x_4 &= 6 \\ -2x_1 - 4x_2 + 7x_3 - x_4 &= 1 \end{aligned} \Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 1 \\ -1 & -1 & 4 & -1 & 6 \\ -2 & -4 & 7 & -1 & 1 \end{array}\right] \xrightarrow[R_3+2R_1]{R_2+R_1}$$

row echelon form
$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 1 \\ 0 & 1 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right] \xrightarrow[R_1+3R_3]{R_2-R_3} \left[\begin{array}{cccc|c} 1 & 2 & 0 & 4 & 10 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right] \xrightarrow{R_1-2R_2}$$

reduced row echelon form
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 6 & 2 \\ 0 & 1 & 0 & -1 & 4 \\ 0 & 0 & 1 & 1 & 3 \end{array}\right]$$

Let $x_4 = \alpha$ "free variable"

$x_3 = 3 - \alpha$

$x_2 = 4 + \alpha$

$x_1 = 2 - 6\alpha$

The set of solutions is $\{(2-6\alpha, 4+\alpha, 3-\alpha, \alpha) : \alpha \in \mathbb{R}\}$
