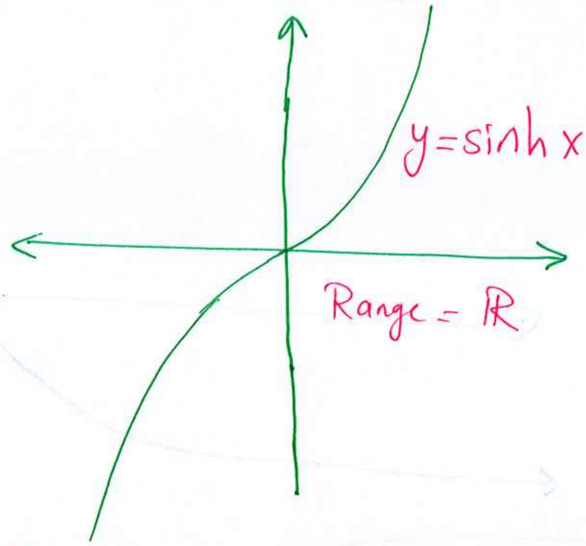
7.7 Hyperbolic Function

Def. The hyperbolic Function are defined as

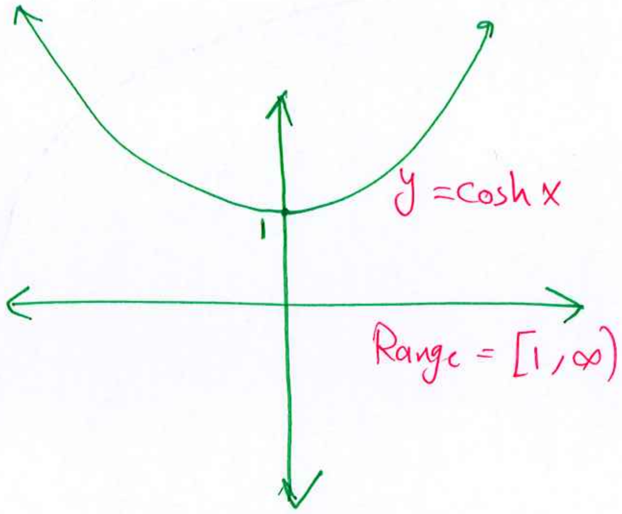
① $\sinh x = \dfrac{e^{x} - e^{-x}}{2}$

② $\cosh x = \dfrac{e^{x} + e^{-x}}{2}$



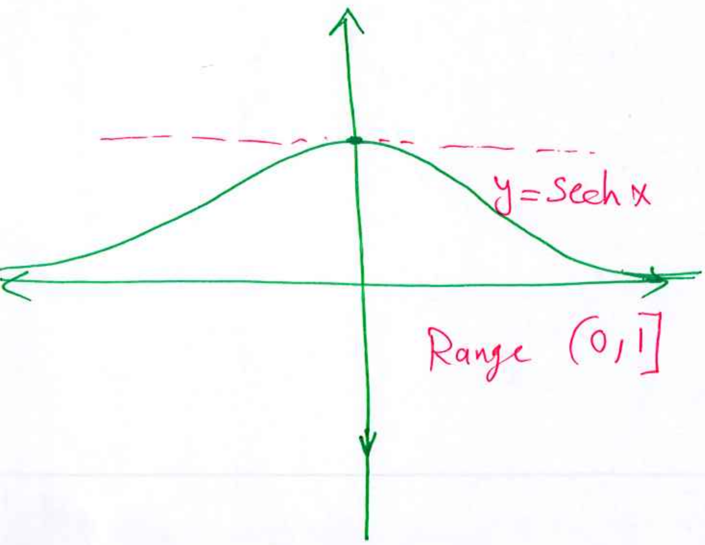
③ $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

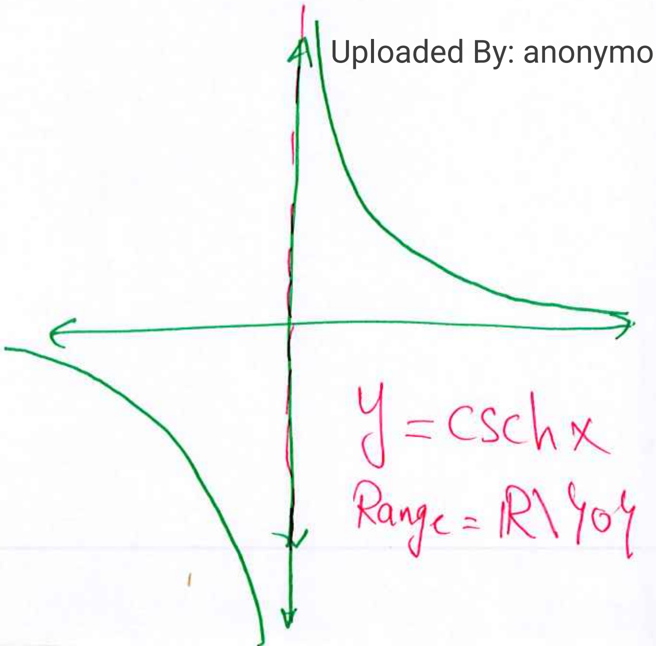
④ $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^{x} + e^{-x}}{e^{x} - e^{-x}}$

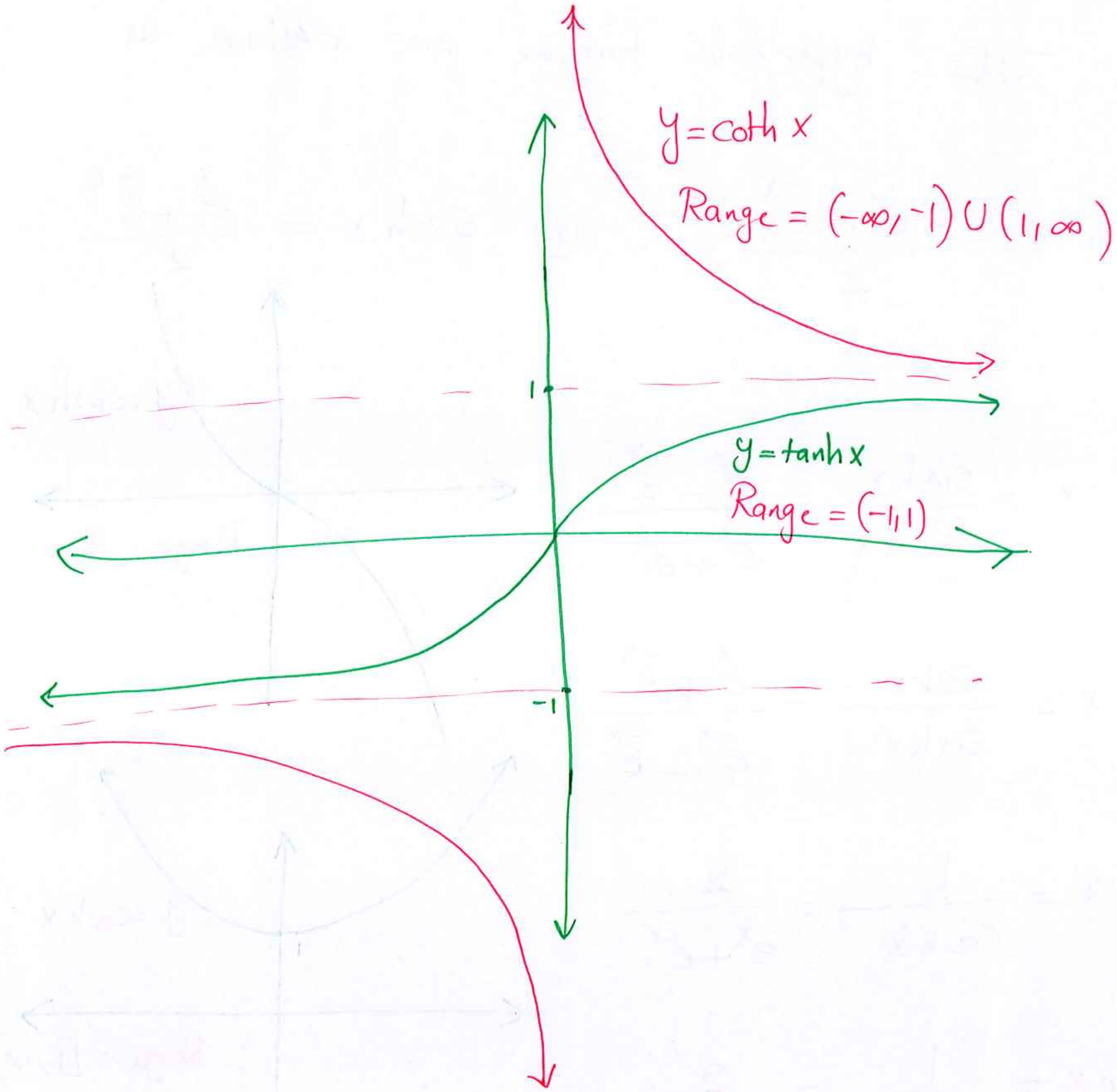


⑤ $\operatorname{sech} x = \dfrac{1}{\cosh x} = \dfrac{2}{e^{x} + e^{-x}}$

⑥ $\operatorname{csch} x = \dfrac{1}{\sinh x} = \dfrac{2}{e^{x} - e^{-x}}$







y = coth x
Range = (-∞, -1) ∪ (1, ∞)
y = tanh x
Range = (-1, 1)
1
-1

Some Identites of Hyperbolic Function:-

* (i) $\cosh^2 x - \sinh^2 x = 1$

(ii) $1 - \tanh^2 x = \operatorname{sech}^2 x$

(iii) $\coth^2 x - 1 = \operatorname{csch}^2 x$

* $\sinh(2x) = 2 \sinh x \cosh x$

* $\cosh(2x) = \cosh^2 x + \sinh^2 x$

$= 2\sinh^2 x + 1$

$= 2\cosh^2 x - 1$

* $\sinh^2 x = \dfrac{\cosh(2x) - 1}{2}$

* $\cosh^2 x = \dfrac{\cosh(2x) + 1}{2}$

* $\cosh x + \sinh x = e^{x}$

$\cosh x - \sinh x = e^{-x}$

* $\cosh x$, $\operatorname{sech} x$ are even functions.

But the others are odd functions.

Derivative of Hyperbolic functions

* $\frac{d}{dx}(\sinh x) = \cosh(x)$

* $\frac{d}{dx}(\sinh u) = \cosh(u) \cdot \frac{du}{dx}$

* $\frac{d}{dx}(\cosh u) = \sinh(u) \cdot \frac{du}{dx}$

* $\frac{d}{dx}(\tanh u) = \operatorname{sech}^2(u) \cdot \frac{du}{dx}$

* $\frac{d}{dx}(\coth u) = -\operatorname{csch}^2(u) \frac{du}{dx}$

* $\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech}(u) \tanh(u) \frac{du}{dx}$

* $\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch}(u) \coth(u) \frac{du}{dx}$

Integrations involving Hyperbolic Functions:—

* $\int \sinh x \, dx = \cosh x + C$
* $\int \cosh x \, dx = \sinh x + C$
* $\int \tanh x \, dx = \ln|\cosh x| + C$
* $\int \coth x \, dx = \ln|\sinh x| + C$
* $\int \operatorname{sech}^2 x \, dx = \tanh x + C$
* $\int \operatorname{csch}^2 x \, dx = -\coth x + C$
* $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + C$
* $\int \operatorname{csch} x \coth x \, dx = -\operatorname{csch} x + C$

Exp: Simplify the following expressions

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① $\sinh(2\ln x) = \dfrac{e^{2\ln x} - e^{-2\ln x}}{2}$

$= \dfrac{x^2 - x^{-2}}{2} = \dfrac{x^4 - 1}{2x^2}$

② $\ln(\cosh x + \sinh x) + \ln(\cosh x - \sinh x) = \ln(e^x) + \ln(e^{-x})$

$= 0$

Exp② If $\cosh x = \dfrac{13}{5}$, $x > 0$

(i) Find tanh x , sech x

$\operatorname{sech} x = \dfrac{5}{13}$

To find tanh x , we use the following identity

$1 - \tanh^2 x = \operatorname{sech}^2 x$

$1 - \tanh^2 x = \dfrac{25}{169}$

$\tanh^2 x = \dfrac{144}{169} \longrightarrow \boxed{\tanh x = \dfrac{12}{13}}$ only positive value because $x > 0$

$\tanh x > 0$

(ii) Find value of x

$\cosh x = \dfrac{e^x + e^{-x}}{2}$

$\dfrac{13}{5} = \dfrac{e^x + e^{-x}}{2}$

$26 = 5e^x + 5e^{-x}$

$5e^{2x} - 26e^x + 5 = 0$

$(5e^x - 1)(e^x - 5)$

$e^x = \frac{1}{5}$ $\qquad e^x = 5$

$\boxed{x = -\ln 5}$ reject $\qquad \boxed{x = \ln 5}$ accept

Find $\frac{dy}{dx}$ for each of the following :-

(1) $y = 2\sqrt{x}\ \tanh(\sqrt{x})$

$$\frac{dy}{dx} = 2\sqrt{x}\ \mathrm{Sech}^2(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} + \tanh(\sqrt{x}) \cdot 2\frac{1}{2\sqrt{x}}$$

$$= \mathrm{Sech}^2(\sqrt{x}) + \frac{\tanh(\sqrt{x})}{\sqrt{x}}$$

(2) $y = \ln(\mathrm{sech}\, x) + \mathrm{Csch}(\ln 3x)$

$$y' = \frac{1}{\mathrm{Sech}\, x} - \mathrm{Sech}\, x \tanh x + -\mathrm{Csch}(\ln 3x) \coth(\ln 3x) \cdot \frac{1}{3x} \cdot 3$$

$$= -\tanh x - \frac{\mathrm{Csch}(\ln 3x)\ \coth(\ln 3x)}{x}$$

(3) $y = \cosh^5\left(\sqrt{e^{3x}+2}\right)$

$$\frac{dy}{dx} = 5\ \cosh^4\left(\sqrt{e^{3x}+2}\right) \cdot \sinh\left(\sqrt{e^{3x}+2}\right) \cdot \frac{3e^{3x}}{2\sqrt{e^{3x}+2}}$$

(4) $y = \sinh(\tan x) + \cot^{-1}(\sinh(x))$

$$\frac{dy}{dx} = \cosh(\tan x) \cdot \sec^2 x + \frac{-1}{1+\sinh^2 x} \cdot \cosh x$$

$$= \sec^2 x\ \cosh(\tan x) - \mathrm{sech}\, x .$$

5 $y = x^{\cosh x}$

$\ln y = \cosh x \ \ln x$

$\frac{y'}{\cancel{y}} \cdot \cancel{y} = \left[\frac{\cosh x}{x} + \sinh x \cdot \ln x \right] x^{\cosh x}$

$y' = \left[\frac{\cosh x}{x} + \sinh x \ \ln x \right] x^{\cosh x}$

Evaluate the following integral :-

① $\int \operatorname{sech} x \, dx = \int \frac{2}{e^x + e^{-x}} \left(\frac{e^x}{e^x}\right) dx$

$\int \frac{2e^x}{e^{2x}+1} dx$ $\qquad u = e^x$, $du = e^x \, dx$

$\int \frac{2 \, du}{u^2+1} = 2 \int \frac{1}{u^2+1} du = 2 \tan^{-1}(u) + C$

$= 2 \tan^{-1}(e^x) + C$

② $\int e^{-x} \cosh x \, dx = \int e^{-x} \left(\frac{e^x + e^{-x}}{2} \right) dx$

$= \frac{1}{2} \int 1 + e^{-2x} \, dx$

$= \frac{1}{2} \left[x + \frac{e^{-2x}}{-2} \right] + C = \frac{1}{2} x - \frac{1}{4} e^{-2x} + C$

③ $\int \operatorname{sech}^3 \tanh x \, dx = \int \operatorname{sech}^2 x \operatorname{sech} x \tanh x \, dx$ $\boxed{165}$

$= -\int u^2 \, du$ $\quad u = \operatorname{sech} x$

$du = -\operatorname{sech} x \tanh x \, dx$

$= -\frac{u^3}{3} + C$

$= \frac{-1}{3} \operatorname{sech}^3 x + C$

④ $\int \frac{1 + \tanh x}{\cosh^2 x} \, dx = \int \frac{1}{\cosh^2 x} \, dx + \int \frac{\sinh x}{\cosh^3 x} \, dx$

$u = \cosh x$

$du = \sinh x \, dx$

$= \int \operatorname{sech}^2 x \, dx + \int u^{-3} \, du$

$= \tanh x - \frac{1}{2} u^{-2} + C$

$= \tanh x - \frac{1}{2 (\cosh x)^2} + C$

⑤ $\int \frac{\sinh 2x}{1 + \cosh x} \, dx = \int \frac{2 \sinh x \cosh x}{1 + \cosh x} \, dx$

$u = 1 + \cosh x$

$du = \sinh x \, dx$

$= \int \frac{2(u-1)}{u} \, du$

$= 2 \left[u - \ln|u| \right] + C = 2 \left[1 + \cosh x - \ln|1 + \cosh x| \right]$

$= 2 \cosh x - 2 \ln|1 + \cosh x| + C$