Exp sketch the graph of $f(x) = \frac{x^2}{x^2-1}$, $f' = \frac{-2x}{(x^2-1)^2}$

$f'' = \frac{6x^2+2}{(x^2-1)^3}$

• $D(f) = \mathbb{R} \setminus \{\pm 1\}$

• H. Asy: $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{x^2-1} = 1$ $\Rightarrow y=1$ is H. Asy.

$\Downarrow$

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2}{x^2-1} = 1$ No O. Asy.

• V. Asy: check $\boxed{x=1}$

$\lim_{x\to1^+} \frac{x^2}{x^2-1} = \frac{1}{\text{small}+} = +\infty$ $\Rightarrow x=1$ is V. Asy.

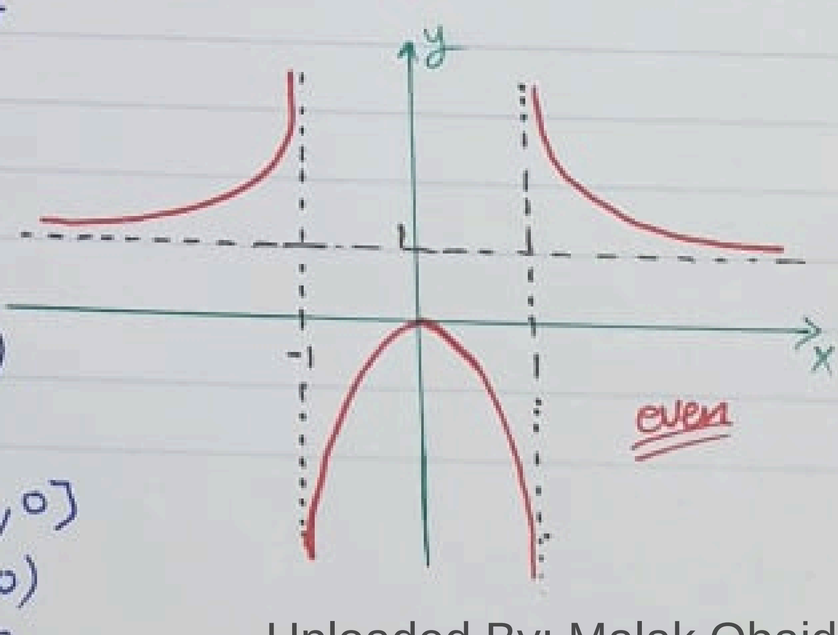
$\lim_{x\to1^-} \frac{x^2}{x^2-1} = \frac{1}{\text{small}-} = -\infty$ ✓

check $\boxed{x=-1}$

$\lim_{x\to-1^+} \frac{x^2}{x^2-1} = \frac{1}{\text{small}-} = -\infty$ $\Rightarrow x=-1$ is V. Asy.

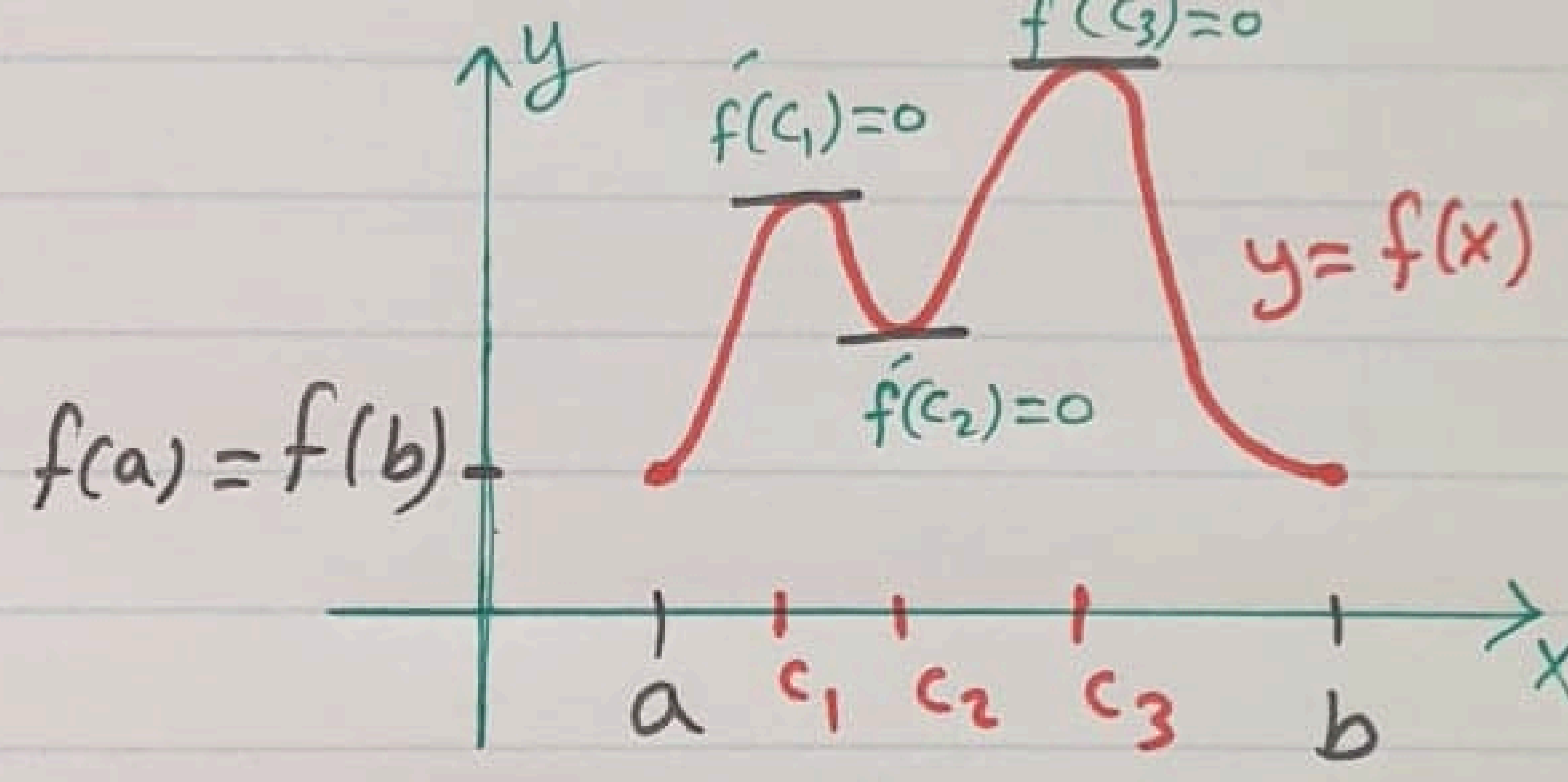
$\lim_{x\to-1^-} \frac{x^2}{x^2-1} = \frac{1}{\text{small}+} = +\infty$ ✓

• f has no inflection points
• Keypoint $(0,0)$
• critical point at $(0,0)$
• $R(f) = (-\infty, 0] \cup (1, \infty)$
• f concave up on $(-\infty,-1) \cup (1,\infty)$
• f concave down on $(-1,1)$
• f is increasing on $(-\infty,-1) \cup (-1,0]$
• f is decreasing on $[0,1) \cup (1,\infty)$
• f has local max of 0 at $x=0$



Th (Rolle's Theorem)

If f is cont. on $[a,b]$
and f is diff on (a,b) s.t $f(a)=f(b)$
then $\exists$ at least one point $c \in (a,b)$ s.t $f'(c)=0$



Exp Let $f(x) = \sin x + 2$

1) Does f have any horizontal tangent on $[0, 2\pi]$

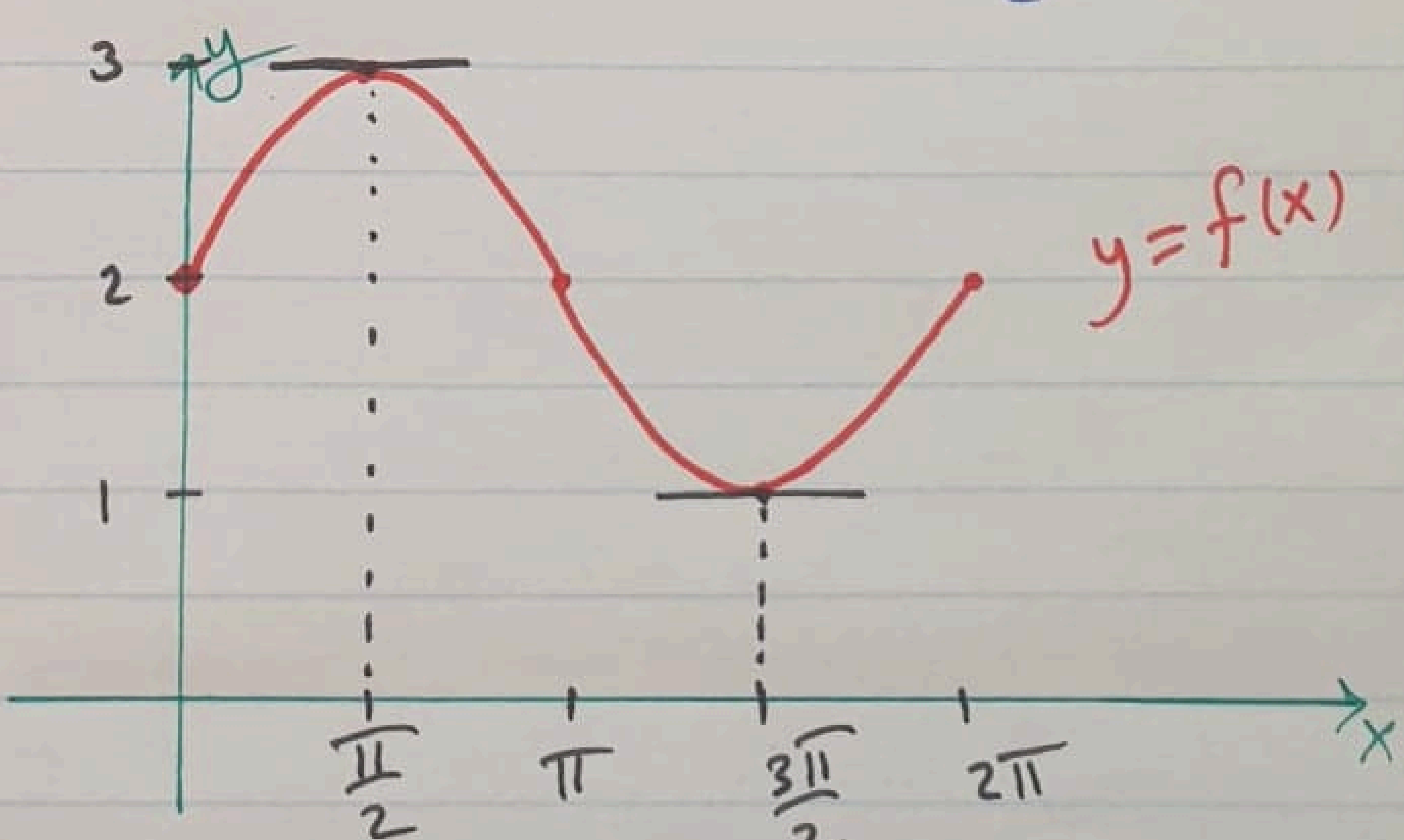
- f is cont. on $[0,2\pi]$
- f is diff on $(0,2\pi)$ $\Rightarrow f'(x) = \cos x$
- $f(0) = 2 = f(2\pi)$

Conditions of Rolle's Th. hold $\Rightarrow \exists\, c \in (0,2\pi)$
s.t $f'(c) = 0$ so f has horizontal tangent at c

2) Find points where f has horizontal tangents.

$f'(c) = 0 \iff \cos c = 0 \iff c = \frac{\pi}{2}$ or $c = \frac{3\pi}{2}$ ✓

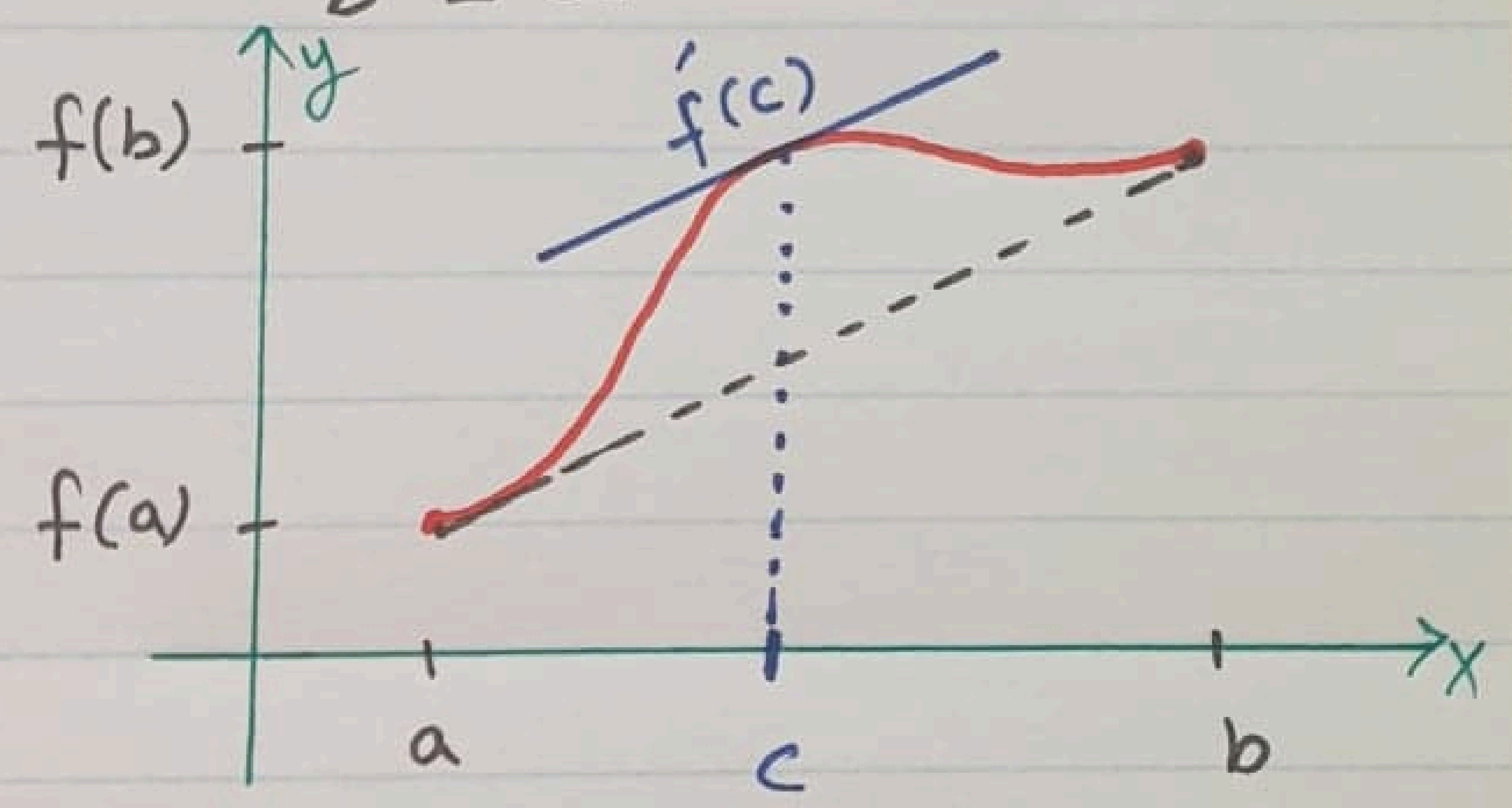
3) sketch $f(x)$



Th (Mean Value Theorem - MVT)

If f is cont. on $[a,b]$ and f is diff on (a,b) then $\exists$ at least one point $c \in (a,b)$ s.t

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



Exp Find the constant c that satisfies the MVT for $f(x) = x^2$ on $[1,3]$

- f is cont. on $[1,3]$
  $f' = 2x$ diff on $(1,3)$
  $\Rightarrow$
  $\exists\, c \in (a,b) = (1,3)$ s.t $\quad f'(c) = \dfrac{f(3)-f(1)}{3-1}$

$$2c = \frac{9-1}{2} = \frac{8}{2} = 4$$

$2c = 4 \Rightarrow c = 2$

Exp Find a, m, b if $f(x) = \begin{cases} 3 & , x = 0 \\ -x^2+3x+a & , 0<x<1 \\ mx+b & , 1 \le x \le 2 \end{cases}$ satisfies the MVT on $[0,2]$

- f cont. at $x=1 \Rightarrow \lim\limits_{x\to1^+} f(x) = \lim\limits_{x\to1^-} f(x)$

  $m+b = -1+3+a \Rightarrow m+b = 2+a$

- f cont at $x=0 \Rightarrow \lim\limits_{x\to0^+} f(x) = f(0)$

  $a = 3$

$\boxed{m+b=5}$

- $f'(x) = \begin{cases} 0 & , x=0 \\ -2x+3 & , 0<x<1 \\ m & , 1<x<2 \end{cases} \Rightarrow f'_+(1) = f'_-(1)$

  $m = -2+3$

  $m = 1$

$b = 4$

Exp Show that $x^3 + 3x + 1 = 0$ has exactly one real root

Take $[a,b] = [-1,0]$

- $f(x) = x^3 + 3x + 1$ is cont. on $[-1,0]$

$f(-1) = -1 - 3 + 1 < 0$

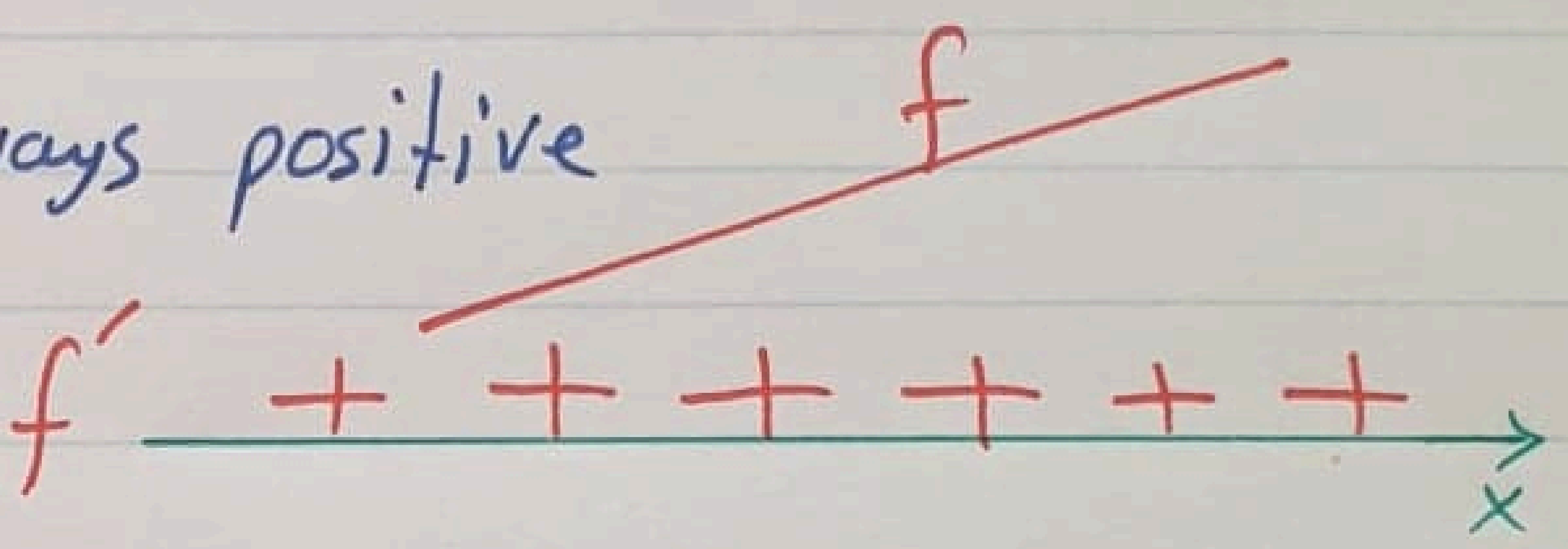
$f(0) = 0 + 0 + 1 > 0$ $\Rightarrow$ By Bolzano Th. $\exists\, c \in (-1,0)$ s.t $f(c) = 0$

c is called root of $f(x)$

- To show it is the only root $\Rightarrow$

$f'(x) = 3x^2 + 3$ is always positive

$f(x)$ is increasing on $\mathbb{R}$



f cross x-axis only at $x = c$

$\Rightarrow$ $x = c$ is the only root