

Find the following Limits

a) $\lim_{t \rightarrow -1} \frac{t^2 + 3t + 2}{t^2 - t - 2} \quad \left(\frac{0}{0} \right)$

$$\begin{aligned} \lim_{t \rightarrow -1} \frac{(t+2)(t+1)}{(t+1)(t-2)} &= \lim_{t \rightarrow -1} \frac{t+2}{t-2} \\ &= \frac{1}{-3} \end{aligned}$$

b) $\lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{1 - x} = \left(\frac{0}{0} \right)$

$$= \lim_{x \rightarrow 1} \frac{1 - \sqrt{x}}{(1 - \sqrt{x})(1 + \sqrt{x})} = \lim_{x \rightarrow 1} \frac{1}{1 + \sqrt{x}} = \frac{1}{2}$$

c) $\lim_{\theta \rightarrow 1} \frac{\theta^4 - 1}{\theta^3 - 1} \quad \left(\frac{0}{0} \right)$

$$= \lim_{\theta \rightarrow 1} \frac{(\theta^2 - 1)(\theta^2 + 1)}{(\theta - 1)(\theta^2 + \theta + 1)} = \lim_{\theta \rightarrow 1} \frac{(\theta - 1)(\theta + 1)(\theta^2 + 1)}{(\theta - 1)(\theta^2 + \theta + 1)}$$

$$= \lim_{\theta \rightarrow 1} \frac{(\theta + 1)(\theta^2 + 1)}{\theta^2 + \theta + 1} = \frac{2 \cdot 2}{3}$$

$\frac{4}{3}$

$$\textcircled{d} \lim_{\theta \rightarrow 0} \frac{\sin 2\theta}{3\theta} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{\theta \rightarrow 0} \frac{\sin 2\theta}{3\theta} = \frac{2}{3}$$

$$\textcircled{e} \lim_{\theta \rightarrow 0} \frac{1 - \cos \theta}{\sin 2\theta} \quad \left(\frac{0}{0}\right)$$

$$\lim_{\theta \rightarrow 0} \frac{(1 - \cos \theta)(1 + \cos \theta)}{2 \sin \theta \cos \theta (1 + \cos \theta)}$$

$$= \lim_{\theta \rightarrow 0} \frac{\sin \theta \cdot \sin \theta}{2 \sin \theta \cos \theta (1 + \cos \theta)}$$

$$= \lim_{\theta \rightarrow 0} \frac{\sin \theta}{2 \cos \theta (1 + \cos \theta)}$$

$$\frac{0}{2(1)(2)} = \frac{0}{4} = 0$$

$$\textcircled{f} \lim_{x \rightarrow \infty} \frac{1 + \sqrt{x}}{1 - \sqrt{x}}$$

$$\lim_{x \rightarrow \infty} \frac{\sqrt{x} \left[\frac{1}{\sqrt{x}} + 1 \right]}{\sqrt{x} \left[\frac{1}{\sqrt{x}} - 1 \right]} = \lim_{x \rightarrow \infty} \frac{\frac{1}{\sqrt{x}} + 1}{\frac{1}{\sqrt{x}} - 1} = \frac{0 + 1}{0 - 1} = -1$$

9 $\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2+1}}{x+1}$

Notes 3

$$\sqrt{x^2} = |x|$$

$$(\sqrt{x})^2 = x$$

$$\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2} \left(\sqrt{1 + \frac{1}{x^2}} \right)}{x \left[1 + \frac{1}{x} \right]} = \lim_{x \rightarrow -\infty} \frac{|x| \sqrt{1 + \frac{1}{x^2}}}{x \left[1 + \frac{1}{x} \right]}$$

$$= \lim_{x \rightarrow -\infty} \frac{-x \sqrt{1 + \frac{1}{x^2}}}{x \left(1 + \frac{1}{x} \right)}$$

$$= \lim_{x \rightarrow -\infty} \frac{-\sqrt{1 + \frac{1}{x^2}}}{1 + \frac{1}{x}} = \frac{-1}{1} = -1$$

10 $\lim_{x \rightarrow -\infty} \frac{\sqrt[3]{x} - \sqrt[5]{x}}{\sqrt[3]{x} + \sqrt[5]{x}}$

$$\lim_{x \rightarrow -\infty} \frac{x^{\frac{1}{3}} - x^{\frac{1}{5}}}{x^{\frac{1}{3}} + x^{\frac{1}{5}}}$$

$$= \lim_{x \rightarrow -\infty} \frac{x^{\frac{1}{3}} \left[1 - x^{\frac{1}{5} - \frac{1}{3}} \right]}{x^{\frac{1}{3}} \left[1 + x^{\frac{1}{5} - \frac{1}{3}} \right]}$$

$$= \lim_{x \rightarrow -\infty} \frac{1 - x^{-\frac{2}{5}}}{1 + x^{-\frac{2}{5}}}$$

$$= \lim_{x \rightarrow -\infty} \frac{1 - \frac{1}{x^{\frac{2}{5}}}}{1 + \frac{1}{x^{\frac{2}{5}}}} = 1$$

$$\textcircled{1} \lim_{k \rightarrow 3^+} \frac{\lfloor k \rfloor}{k}$$

$$\frac{3}{3} = 1$$

$$\textcircled{1} \lim_{x \rightarrow 0} x \sin \frac{1}{x} \quad 0 \cdot \sin(\infty)$$

$$-1 \leq \sin \frac{1}{x} \leq 1$$

$$x \rightarrow 0^+$$

$$-x \leq x \sin \frac{1}{x} \leq x \cdot 1$$

$$x \leq x \sin \frac{1}{x} \leq x$$

take limit to all sides

$$\lim_{x \rightarrow 0^+} x \leq \lim_{x \rightarrow 0^+} x \sin \frac{1}{x} \leq \lim_{x \rightarrow 0^+} x$$

$$0 \leq \lim_{x \rightarrow 0^+} x \sin \frac{1}{x} \leq 0$$

same thing when $x \rightarrow 0^-$

$$x \rightarrow 0^- \quad -1 \leq \sin \frac{1}{x} \leq 1$$

$$-x > x \sin \frac{1}{x} \geq -x$$

$\lfloor k \rfloor$ Greatest integer function

$$\lfloor 1.3 \rfloor = 1$$

$\lceil k \rceil$ = smallest integer function

$$\lceil 1.3 \rceil = 2$$

Sandwich Theorem

$$\textcircled{1} g(x) \leq f(x) \leq h(x)$$

$$\textcircled{2} \lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = \alpha$$

$$\text{then } \lim_{x \rightarrow c} f(x) = \alpha$$

2 Find the asymptotes of the following functions
then sketch their graphs.

1 $f(x) = \frac{x+1}{x-1}$

Horizontal asymptotes

"same as $x \rightarrow -\infty$ "

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{x+1}{x-1} = 1$$

$$\boxed{y=1}$$

No oblique asymptotes

Vertical asymptotes

$$x-1=0 \rightarrow x=1$$

$$\lim_{x \rightarrow 1^+} \frac{x+1}{x-1} = +\infty$$

$$\lim_{x \rightarrow 1^-} \frac{x+1}{x-1} = -\infty$$

$\boxed{x=1}$ vertical asymptote

x -int

$$y=0$$

$$0 = \frac{x+1}{x-1}$$

$$x+1=0 \rightarrow x=-1$$

$$(-1, 0)$$

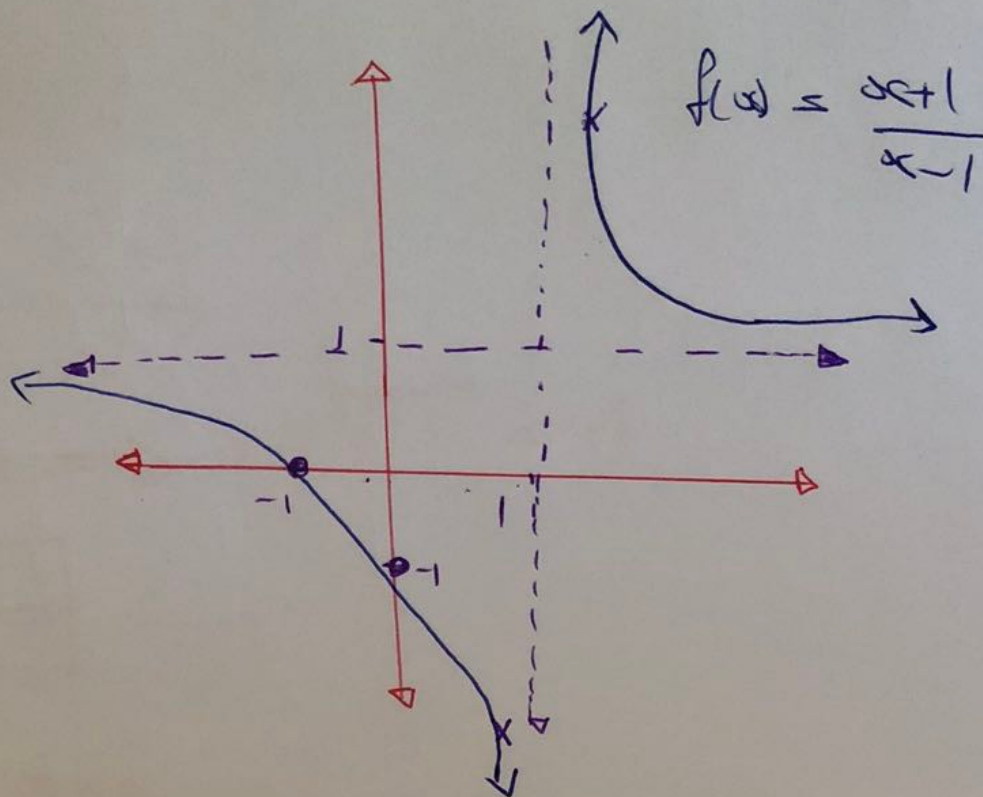
~~y~~ -int

~~y~~

$$x=0$$

$$y = \frac{1}{-1} = -1$$

$$(0, -1)$$



⑥ $y = \frac{x^3+1}{x^2}$

No Horizontal Asymptote

Oblique asymptote

$$y = x$$

$$\begin{array}{r} x \\ x^2 \overline{) x^3 + 1} \\ \underline{x^3} \\ 1 \end{array}$$

Vertical asymptote

$$x + \frac{1}{x^2}$$

$$x^2 = 0$$

$$x = 0$$

$$\lim_{x \rightarrow 0^+} \frac{x^3+1}{x^2} = +\infty \quad \text{and} \quad \lim_{x \rightarrow 0^-} \frac{x^3+1}{x^2} = +\infty$$

$$x = 0$$

Vertical asymptote

x-int

$$y=0$$

$$0 = \frac{x^3 + 1}{x^2}$$

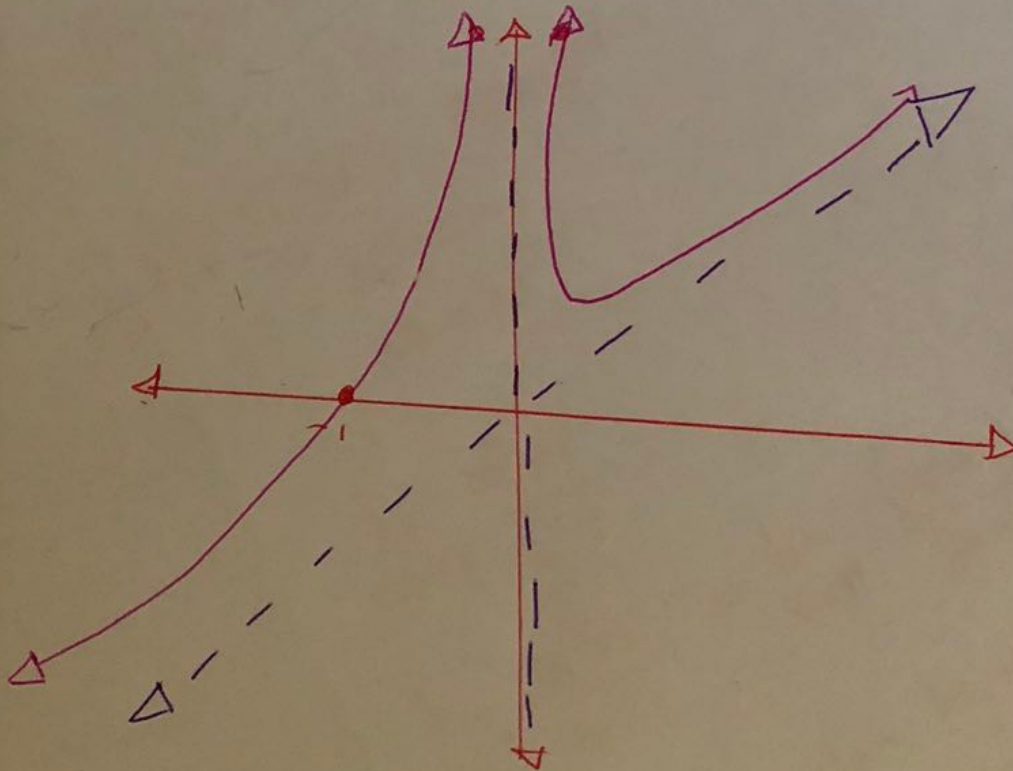
$$x^3 = -1 \Rightarrow x = \sqrt[3]{-1} = -1$$

$$\boxed{(-1, 0)}$$

y-int, $x=0$

$$y = \frac{x^3 + 1}{x^2}$$

NO y-int



$$\textcircled{c} f(x) = \frac{x^2+1}{x-1}$$

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No Horizontal asymptote. ✓

Oblique asymptote

$$y = x+1 \quad \checkmark$$

Vertical Asymptote

$$x=1$$

$$\lim_{x \rightarrow 1^+} \frac{x^2+1}{x-1} = +\infty$$

$$\lim_{x \rightarrow 1^-} \frac{x^2+1}{x-1} = -\infty$$

so $x=1$ Vertical Asymptote

$$x\text{-int} \quad y=0$$

$$0 = \frac{x^2+1}{x-1}$$

$$\rightarrow x^2+1=0$$

No x-int

$$\begin{array}{r} x+1 \\ x-1 \overline{) x^2+1} \\ \underline{-(x^2-x)} \\ 2x+1 \end{array}$$

$$\textcircled{-} \frac{x+1}{x-1}$$

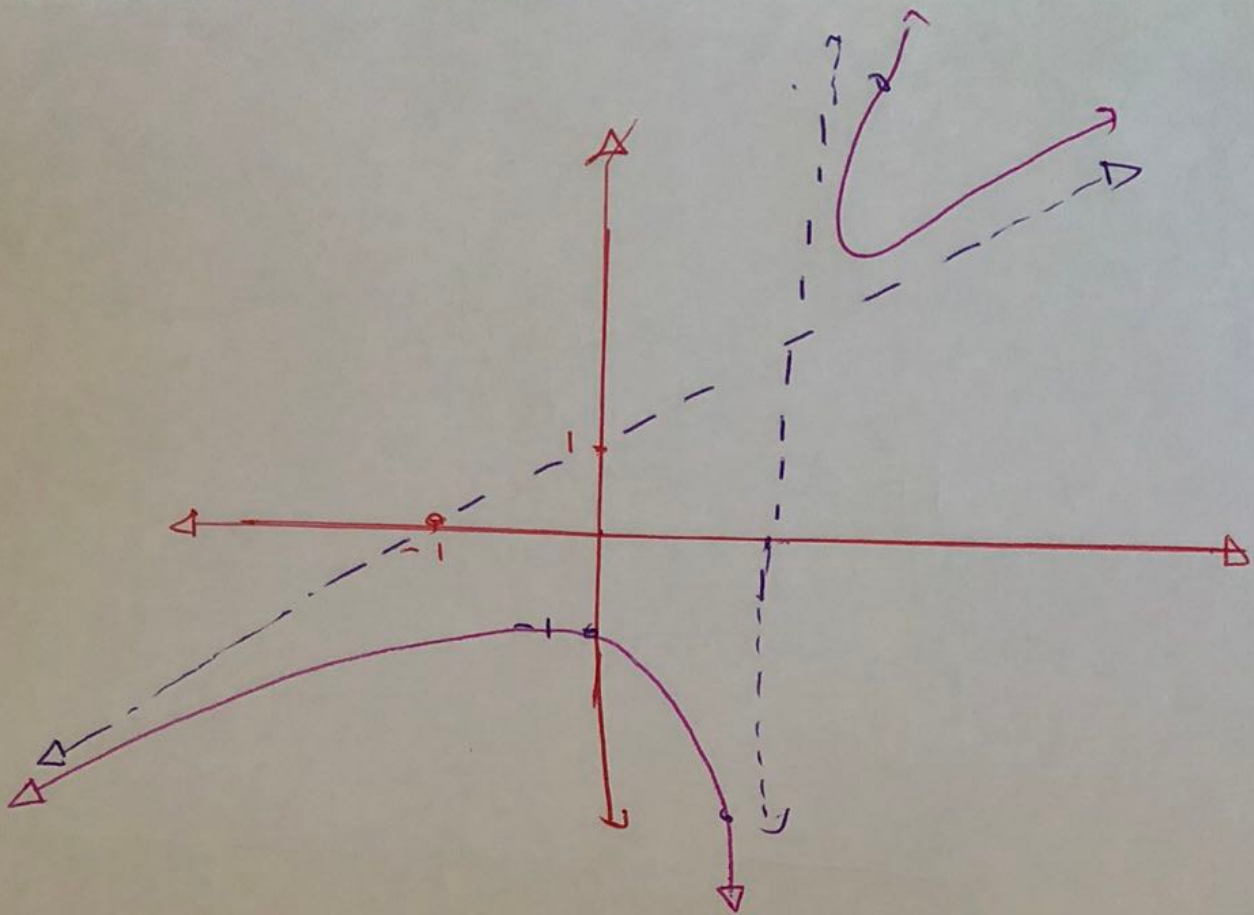
$$2$$

$$y\text{-int} \quad x=0$$

$$y = \frac{0+1}{0-1} = -1$$

$$(0, -1)$$

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$$d) f(x) = \frac{x^3+1}{x^2-1}$$

(10)

① No Horizontal asymptote:

② $y = x$ Oblique Asymptote

$$\begin{array}{r} x \\ x^2-1 \overline{) x^3+1} \\ \underline{-(x^3-x)} \\ x+1 \end{array}$$

③ Vertical Asymptotes

$$x^2-1=0$$

$$x=\pm 1$$

$$\lim_{x \rightarrow 1} \frac{x^3+1}{x^2-1}$$

$$\lim_{x \rightarrow 1^+} \frac{x^3+1}{x^2-1} = \lim_{x \rightarrow 1^+} \frac{(x+1)(x^2-x+1)}{(x+1)(x-1)} = +\infty$$

$$\lim_{x \rightarrow 1^-} \frac{x^3+1}{x^2-1} = \lim_{x \rightarrow 1^-} \frac{(x+1)(x^2-x+1)}{(x+1)(x-1)} = -\infty$$

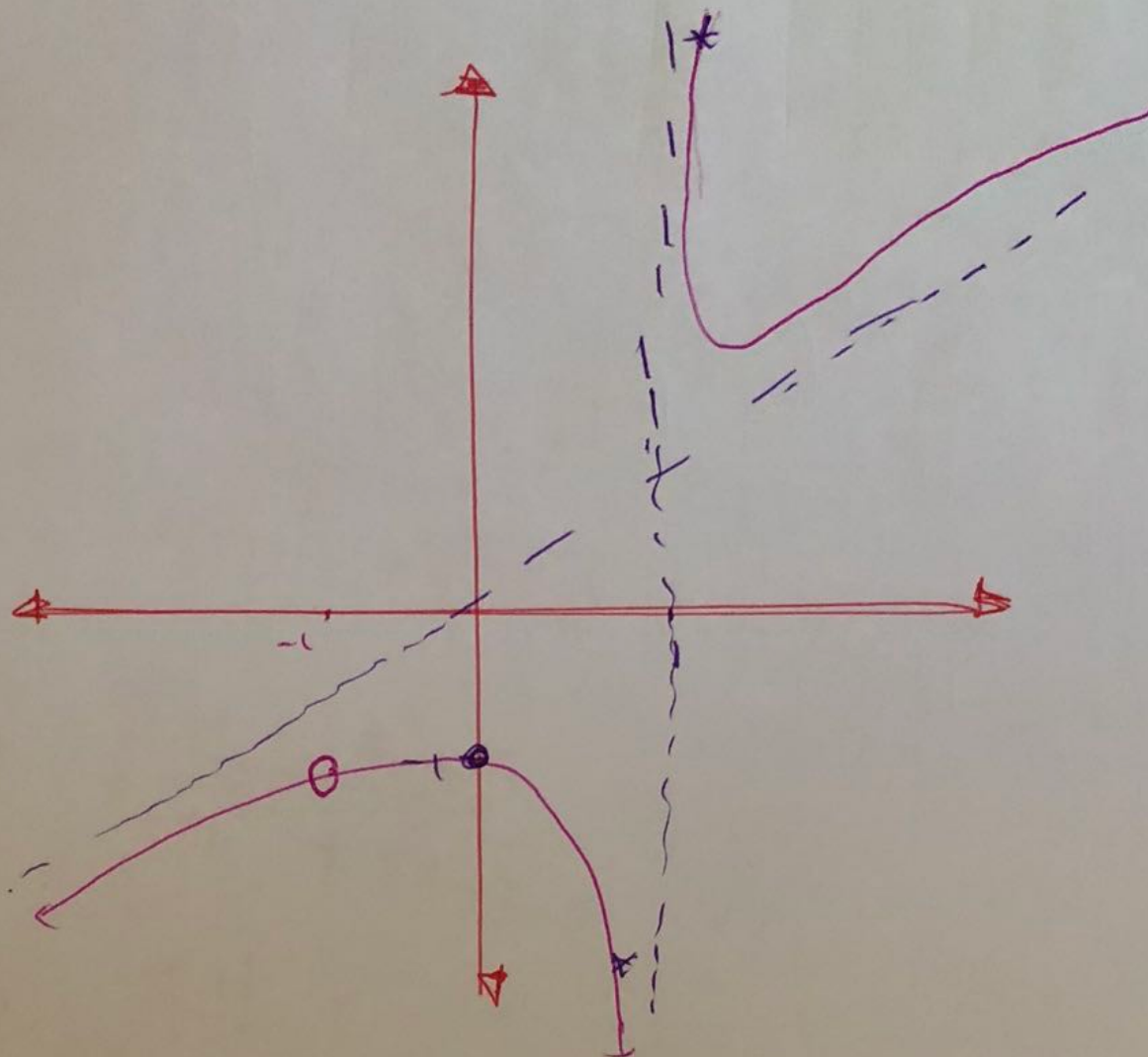
$$\boxed{x=1} \quad \text{V.A.}$$

$$\lim_{x \rightarrow -1} \frac{x^3+1}{x^2-1}$$

$$\lim_{x \rightarrow -1^+} \frac{x^3+1}{x^2-1} = \lim_{x \rightarrow -1^+} \frac{(x+1)(x^2-x+1)}{(x+1)(x-1)} \stackrel{(0/0)}{=} \lim_{x \rightarrow -1^+} \frac{x^2-x+1}{x-1} = \frac{3}{-2}$$

$$\lim_{x \rightarrow -1^-} \frac{x^3+1}{x^2-1} = \frac{3}{-2}$$

$$\boxed{x=-1 \text{ is not a V.A.}}$$



x-int

$$y=0$$

$$0 = \frac{x^3 + 1}{x^2 - 1}$$

$$x^3 + 1 = 0$$

$$x = -1$$

No x-int

y-int

$$x=0$$

$$y = -1$$

$$(0, -1)$$

$$(1) \lim_{x \rightarrow \infty} \sqrt{x^2+1} - \sqrt{x^2-x}$$

$$\lim_{x \rightarrow \infty} \frac{[\sqrt{x^2+1} - \sqrt{x^2-x}][\sqrt{x^2+1} + \sqrt{x^2-x}]}{\sqrt{x^2+1} + \sqrt{x^2-x}}$$

$$= \lim_{x \rightarrow \infty} \frac{\left(\sqrt{x^2+1}\right)^2 - \left(\sqrt{x^2-x}\right)^2}{\sqrt{x^2+1} + \sqrt{x^2-x}}$$

$$= \lim_{x \rightarrow \infty} \frac{x^2+1 - x^2+x}{\sqrt{x^2+1} + \sqrt{x^2-x}}$$

$$= \lim_{x \rightarrow \infty} \frac{x+1}{\sqrt{x^2+1} + \sqrt{x^2-x}}$$

$$= \lim_{x \rightarrow \infty} \frac{x \left[1 + \frac{1}{x}\right]}{\sqrt{x^2} \left[\sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x}}\right]}$$

$$\begin{aligned} x &\rightarrow \infty \\ \sqrt{x^2} &\Rightarrow |x| \end{aligned}$$

$$= \lim_{x \rightarrow \infty} \frac{1 + \frac{1}{x}}{\sqrt{1 + \frac{1}{x^2}} + \sqrt{1 - \frac{1}{x}}} = \frac{1}{1+1} = \frac{1}{2}$$