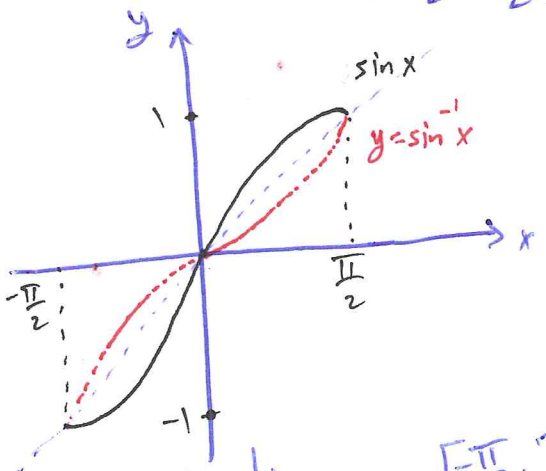
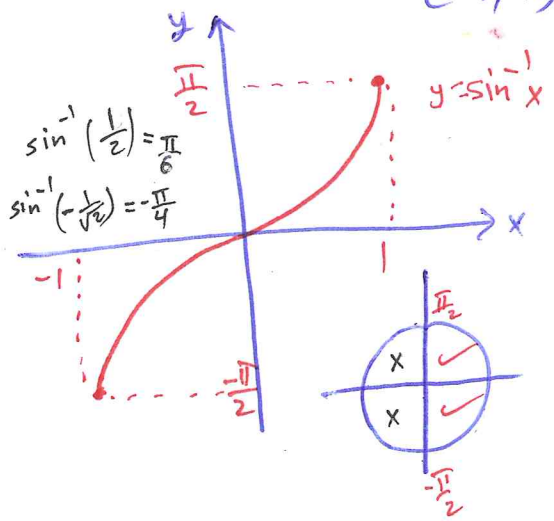
7.6 Inverse Trigonometric Functions

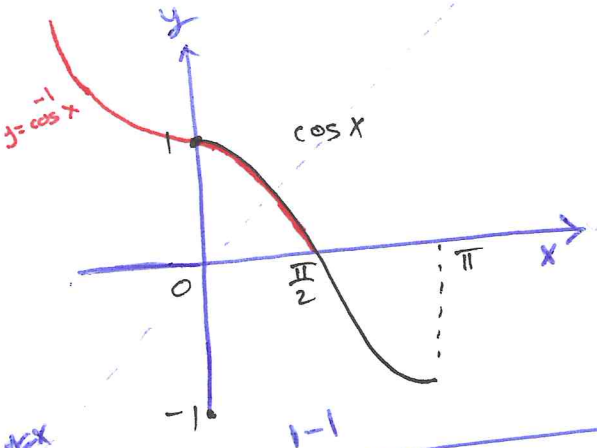
If $f(x)=\sin x$ on $[-\frac{\pi}{2},\frac{\pi}{2}]$, then $f^{-1}(x)=y=\sin^{-1}x$ $=\arcsin x$ on $[-1,1]$



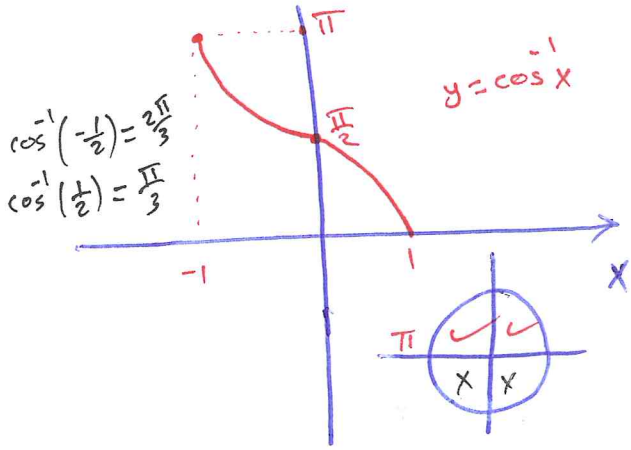
1-1 function on $[-\frac{\pi}{2},\frac{\pi}{2}]$



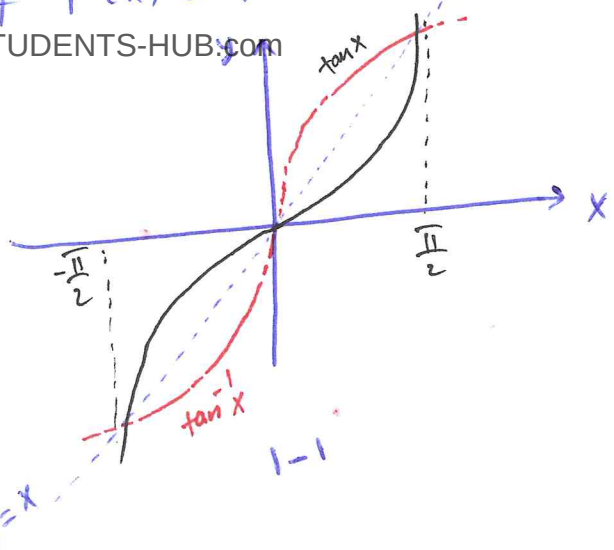
If $f(x)=\cos x$ on $[0,\pi]$, then $f^{-1}(x)=y=\cos^{-1}x$ on $[-1,1]$ $=\arccos x$



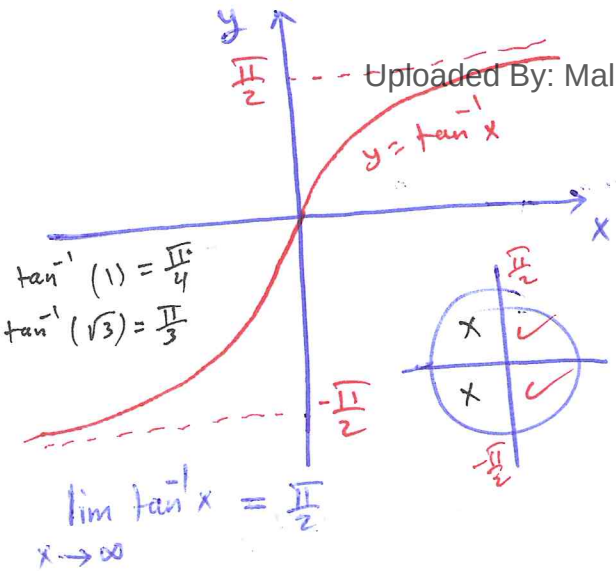
1-1



If $f(x)=\tan x$ on $(-\frac{\pi}{2},\frac{\pi}{2})$, then $f^{-1}(x)=y=\tan^{-1}x$ on $\mathbb{R}$ $=\arctan x$



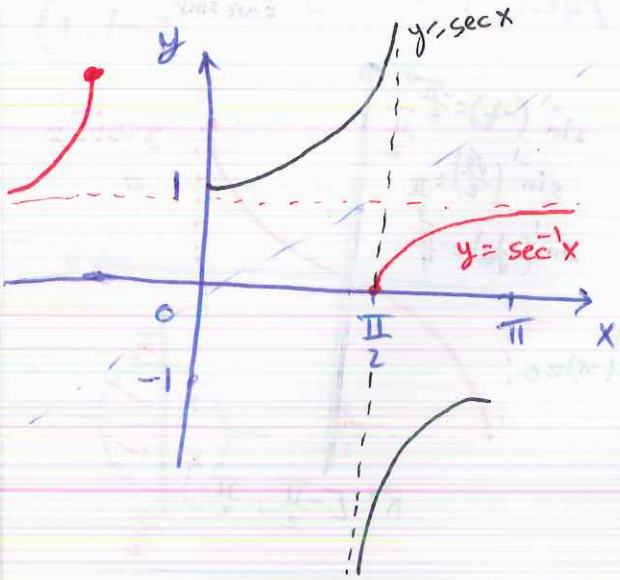
1-1

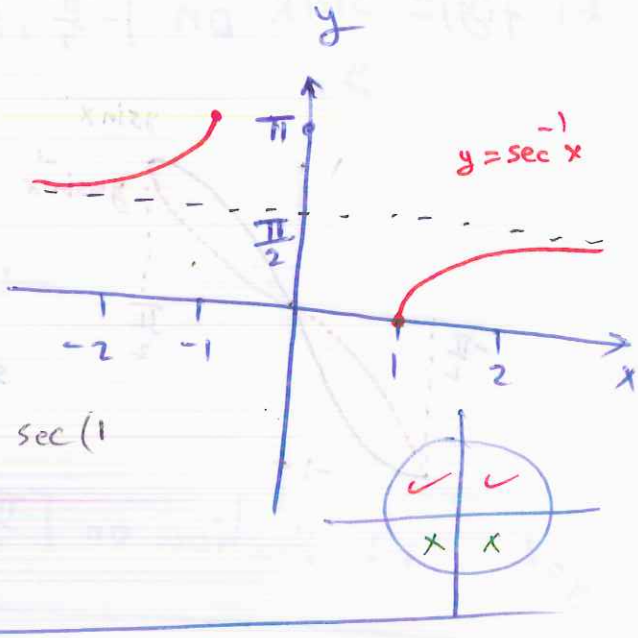


$\lim_{x\to\infty}\tan^{-1}x=\frac{\pi}{2}$

• If $f(x) = \sec x$ on $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, then

$f^{-1}(x) = \sec^{-1} x$ on $(-\infty, -1] \cup [1, \infty)$

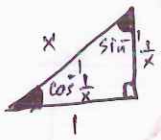




Note that

$\csc^{-1}(-2) = \sin^{-1}(-\frac{1}{2}) = -\frac{\pi}{6}$

$\sec^{-1} x = \cos^{-1}(\frac{1}{x}) = \frac{\pi}{2} - \sin^{-1}(\frac{1}{x})$

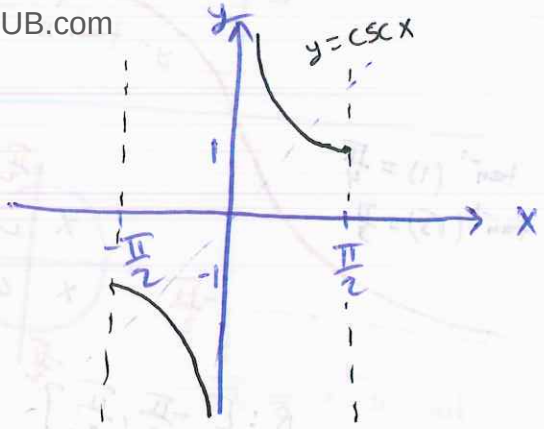


$\csc^{-1} x = \sin^{-1}(\frac{1}{x}) = \frac{\pi}{2} - \cos^{-1}(\frac{1}{x})$

Thus,

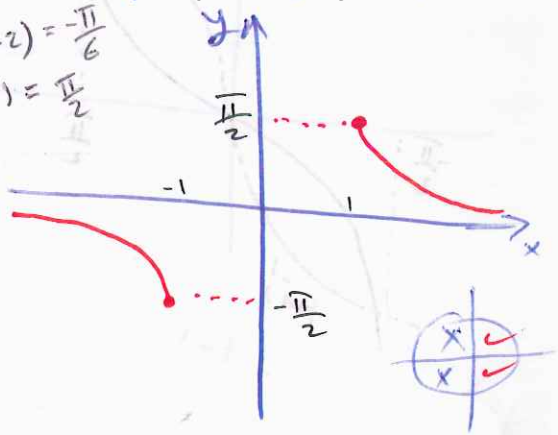
$$\sec^{-1} x + \csc^{-1} x = \cos^{-1}(\frac{1}{x}) + \sin^{-1}(\frac{1}{x}) = \frac{\pi}{2}$$

• If $f(x) = \csc x$ on $[-\frac{\pi}{2}, \frac{\pi}{2}] - \{0\}$, then $f^{-1}(x) = \csc^{-1} x$ on $(-\infty, -1] \cup [1, \infty)$

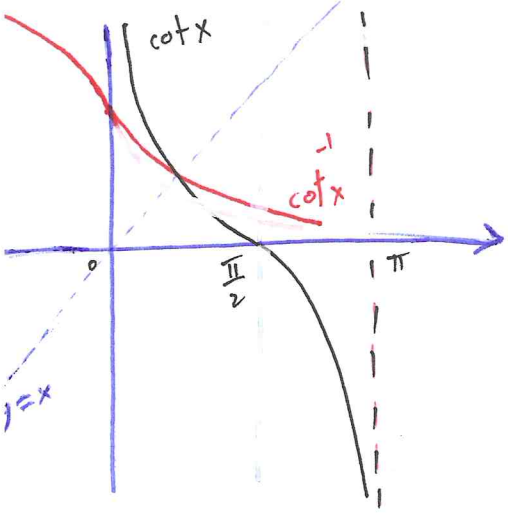


$\csc(-2) = -\frac{\pi}{6}$

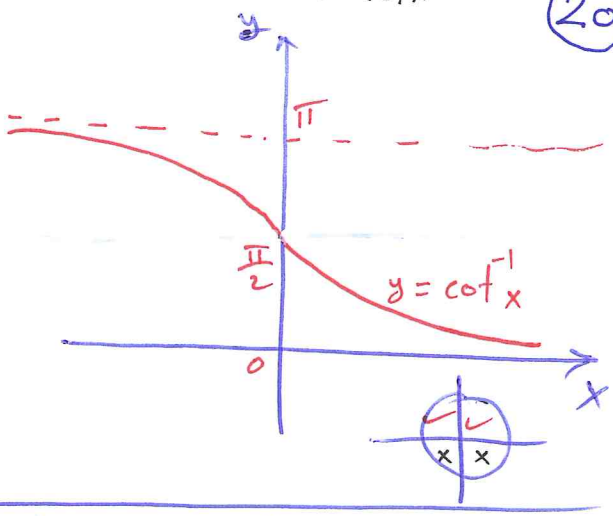
$\csc(1) = \frac{\pi}{2}$

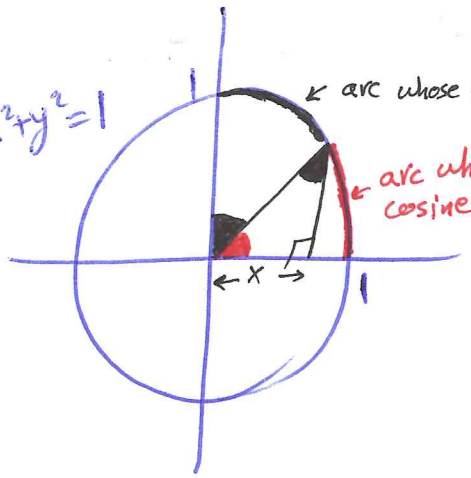


If $f(x) = \cot x$ on $(0, \pi)$, then $f^{-1}(x) = y = \cot^{-1} x$ on $\mathbb{R}$



= arc cot x



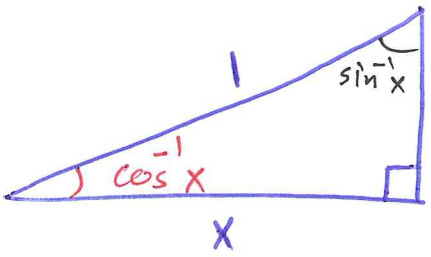


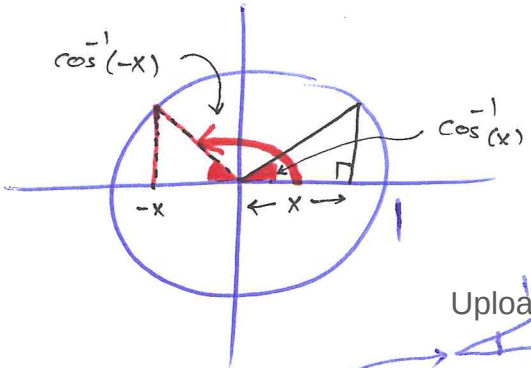
$s = r\theta = \theta$

Note that

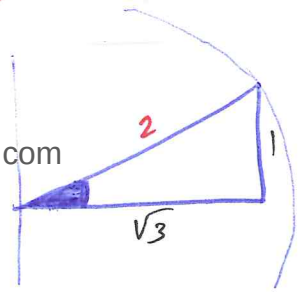
$\cos^{-1} x + \cos^{-1}(-x) = \pi$

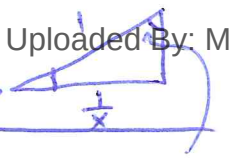
$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$





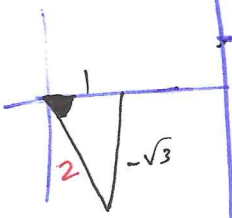
$\tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}$

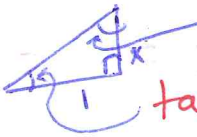




$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \sin^{-1}\left(\frac{1}{x}\right)$

- $\tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$





$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

$\sec^{-1} x + \csc^{-1} x = \frac{\pi}{2}$

* If $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1} x$, Then

$$\frac{df^{-1}}{dx}(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\cos(f^{-1}(x))} = \frac{1}{\cos(\sin^{-1} x)}$$

$$= \frac{1}{\sqrt{1 - \sin^2(\sin^{-1} x)}} = \frac{1}{\sqrt{1 - x^2}}$$

$$\sin(\sin^{-1} x) = x$$

$$\sin^2(\sin^{-1} x) = \sin(\sin^{-1} x)\ \sin(\sin^{-1} x)$$
$$= x \quad x = x^2$$

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1 - u^2}} \frac{du}{dx}, \quad |u| < 1$$

Example $\frac{d}{dx}(\sin^{-1}\sqrt{2}x) = \frac{1}{\sqrt{1 - (\sqrt{2}x)^2}} \sqrt{2} = \frac{\sqrt{2}}{\sqrt{1 - 2x^2}}$

$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1 + u^2} \frac{du}{dx}$$

$f(x) = \tan x$ with $f^{-1}(x) = \tan^{-1} x$

$$\frac{df^{-1}(x)}{dx} = \frac{1}{f'(f^{-1}(x))} = \frac{1}{\sec^2(\tan^{-1} x)} = \frac{1}{1 + \tan^2(\tan^{-1} x)}$$

$$= \frac{1}{1 + x^2}$$

$\tan(\tan^{-1} x) = x$

Example $y = \ln \tan^{-1} x \Rightarrow \frac{dy}{dx} = \frac{1}{\tan^{-1} x}\left(\frac{1}{1 + x^2}\right)$

Similarly

$$\frac{d}{dx}\left(\cos^{-1}u\right) = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx} \quad , |u|<1$$

$$\frac{d}{dx}\left(\cot^{-1}u\right) = \frac{-1}{1+u^2}\frac{du}{dx}$$

$$\frac{d}{dx}\left(\sec^{-1}u\right) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \quad , |u|>1$$

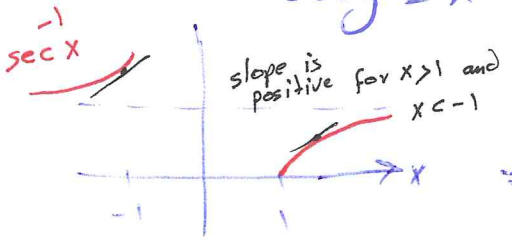
Let $y = \sec^{-1}x \Rightarrow \sec y = x$

$$\Rightarrow \sec y \tan y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

But $\sec y = x$ and $\tan y = \pm\sqrt{\sec^2 y - 1}$

$$= \pm\sqrt{x^2-1}$$



$$\Rightarrow \frac{dy}{dx} = \pm\frac{1}{x\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}$$

Example $y = \sec^{-1}(2x+1) \Rightarrow \frac{dy}{dx} = \frac{2}{|2x+1|\sqrt{(2x+1)^2-1}} = \frac{1}{|2x+1|\sqrt{x^2+x}}$

Similarly

$$\frac{d}{dx}\left(\csc^{-1}u\right) = \frac{-1}{|u|\sqrt{u^2-1}}\frac{du}{dx} \quad , |u|>1$$

Now for any constant $a \neq 0$

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C \quad , \quad u^2 < a^2$$

$$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1}\left|\frac{u}{a}\right| + C \quad , \quad |u| > a > 0$$

Exp. $\int_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x \Big|_{\frac{1}{\sqrt{2}}}^{\frac{\sqrt{3}}{2}} = \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1}\frac{1}{\sqrt{2}}$

$= \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}$

Exp $\int \frac{dx}{\sqrt{4x - x^2}} = \int \frac{dx}{\sqrt{4 - (x-2)^2}}$

$4x - x^2 = -(x^2 - 4x)$

$= -(x-2)^2 + 4$

$= 4 - (x-2)^2$

$a = 2$

$u = x - 2$

$du = dx$

$= \int \frac{du}{\sqrt{a^2 - u^2}}$

$= \sin^{-1}\left(\frac{u}{a}\right) + C$

$= \sin^{-1}\left(\frac{x-2}{2}\right) + C$