

Taylor series & Maclaurin series

- $f(x)$ is a differentiable function for all orders around a center $x=a$

→ The Taylor series is :-

$$T.S = f(a) + \frac{f'(a)(x-a)^1}{1!} + \frac{f^{(2)}(a)(x-a)^2}{2!} + \dots + \frac{f^{(n)}(a)(x-a)^n}{n!} + \dots$$

This is the derivative order

nth derivative at $x=a$

→ The Maclaurin series occurs when $a=0$:-

$$M.S = f(0) + \frac{f'(0)(x)}{1!} + \frac{f^{(2)}(0)(x)^2}{2!} + \dots + \frac{f^{(n)}(0)(x)^n}{n!} + \dots$$

→ To know whether a Taylor series Conv or div ?
 You use the technique in 10.7 (Power series)
 or if it was a Geometric series you use r

• Note :-

The function $f(x)$ equals the Taylor series only in the interval of convergence

Taylor Polynomial :- $T_n(x) = f(a) + \frac{f'(a)(x-a)}{1!} + \dots + \frac{f^{(n)}(a)(x-a)^n}{n!} + \dots$

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Things that you have to know in this section

You should know these T-series by heart:-

$$\rightarrow e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{around } a=0 \quad -\infty < x < \infty$$

$$\rightarrow \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{around } a=0 \quad -\infty < x < \infty$$

$$\rightarrow \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{around } a=0 \quad -\infty < x < \infty$$

$$\rightarrow \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n} \quad \text{where } |x| < 1 \quad \text{around } a=0$$

$$\rightarrow \frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots = \sum_{n=0}^{\infty} (-1)^n x^n \quad -1 < x < 1$$

$$\rightarrow \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n \quad -1 < x < 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots + \frac{(-1)^n x^{2n+1}}{2n+1} + \dots \quad -1 \leq x \leq 1$$

Steps of finding a Taylor series :-

Exp: find the Taylor series generated by $f(x) = \frac{1}{x}$ at $a=2$

Step 1

1- find the first 4 or 5 derivatives

$$f(x) = \frac{1}{x}$$

$$f(2) = \frac{1}{2}$$

negative when $n=1$

$$f'(x) = -\frac{1}{x^2}$$

$$f'(2) = -\frac{1}{4}$$

positive when $n=2$

$$f''(x) = \frac{2}{x^3}$$

$$f''(2) = \frac{2}{8} = \frac{1}{4}$$

so $(-1)^n$

$$f'''(x) = -\frac{2 \cdot 3}{x^4}$$

$$f'''(2) = -\frac{2 \cdot 3}{2^4}$$

$$f^{(4)}(x) = \frac{2 \cdot 3 \cdot 4}{x^5}$$

$$f^{(4)}(2) = \frac{2 \cdot 3 \cdot 4}{2^5}$$

Step 2

$$f^{(n)}(x) = \frac{(-1)^n 2 \cdot 3 \cdot 4 \dots n}{x^{n+1}}$$

$$f^{(n)}(2) = \frac{(-1)^n (2 \cdot 3 \cdot 4 \dots n)}{2^{n+1}} \quad n!$$

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Now: T.S = $f(2) + f'(2) \frac{(x-2)}{1} + f''(2) \frac{(x-2)^2}{2!} + \dots + \frac{f^{(n)}(2)(x-2)^n}{n!} + \dots$

$$= \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{2(x-2)^2}{(2^2 \cdot 2!)} + \dots + \frac{(-1)^n n!}{2^{n+1}} \frac{(x-2)^n}{n!} + \dots$$

$$\text{T.S} = \frac{1}{2} - \frac{(x-2)}{2^2} + \frac{(x-2)^2}{2^3} + \dots + \frac{(-1)^n (x-2)^n}{2^{n+1}} + \dots$$

Know if Conv or div:-

It's a G.S

$$r = -\frac{(x-2)}{2}$$

$$|r| < 1 \rightarrow \text{Conv}$$

$$\text{So } 0 < x < 4$$

Result:- $\frac{1}{x} = \frac{1}{2} - \frac{(x-2)}{2^2} + \dots$ when $0 < x < 4$

• If he asked for the Mac. series of the function $\frac{1}{1+x}$

$$\text{M.S} = -f(0) + f'(0) \frac{x}{1} + f''(0) \frac{x^2}{2!} - \dots + \frac{f^{(n)}(0) x^n}{n!} + \dots$$

Step 1:- $f(x) = \frac{1}{1+x}$

$$f'(x) = \frac{-1}{(1+x)^2}$$

$$f''(x) = \frac{2}{(1+x)^3}$$

$$f'''(x) = \frac{-2 \cdot 3}{(1+x)^4}$$

$$f(0) = 1$$

$$f'(0) = -1$$

$$f''(0) = 2$$

$$f'''(0) = -2 \cdot 3$$

Step 2:- $\text{M.S} = 1 - x + \frac{2x^2}{2!} - \frac{2 \cdot 3 x^3}{3!} + \dots + \frac{(-1)^n n! x^n}{n!} + \dots$

$$\text{M.S} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + \dots$$

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