Continuity الاتصال

cont. : continuous

Def f is cont. at x_0 if

1. $\lim_{x \to x_0} f(x)$ exists and
2. $f(x_0)$ exists and
3. $\lim_{x \to x_0} f(x) = f(x_0)$

$$\lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) = \underline{\underline{L}}$$

Exp $f(x) = x^2 - 3$ cont. at $x = 2$

$f(2) = 2^2 - 3 = 4 - 3 = 1$

$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x^2 - 3)$
$= 4 - 3$
$= 1$

$f(2) = \lim_{x \to 2} f(x)$

$1 = 1$

* f cont. on $[a, b]$ if
f cont. on every point $x_0 \in [a, b]$

Examples for cont. function on $(-\infty, \infty)$

→ $f(x) = x^2 - 4x + 5$ "all polynomials" ندرسها اكثر

$f(x) = |x|$ ✓

$f(x) = e^x$ ✓

$f(x) = \cos x$ ✓

$f(x) = \sin x$ ✓

Exp Rational function $R(x)$ اقتران نسبي

$$R(x) = \frac{f(x)}{g(x)}$$

f, g polynomials
كثيرات حدود

$R(x)$ is cont. everywhere except where $g(x)=0$

Exp $R(x) = \dfrac{x^2 - 9}{x+3}$ is cont. on $\mathbb{R} \setminus \{-3\}$

$R(x) = \dfrac{x}{x^2 - 4}$ is cont. on $\mathbb{R} \setminus \{-2, 2\}$

Exp Assume $f(x) = \begin{cases} ax + b, & x \le 0 \\ x^2 + 3a - b, & 0 < x \le 2 \\ 3x - 5, & 2 < x \end{cases}$

is cont. Find a, b and sketch $f(x)$

f is cont. at $x = 2 \Rightarrow f(2) = \lim_{x \to 2} f(x)$

f is cont. at x = ...

$\Rightarrow \lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x)$

$3(2) - 5 = 2^2 + 3a - b$

$6 - 5 = 4 + 3a - b$

$1 = 4 + 3a - b$

$\boxed{3a = b - 3}$ — (2)

f is cont at $x=0$ $\Rightarrow f(0) = \lim_{x\to 0} f(x)$

$\Rightarrow \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$

$0^2 + 3(a) - b = a(0) + b$

$0 + 3a - b = 0 + b$

$\boxed{2b = 3a}$ — (1)

$2b = b - 3$

$\boxed{b = -3} \Rightarrow a = -2$

f cont at x_0

$\Rightarrow \lim_{x\to x_0^+} f(x_0) = \lim_{x\to x_0^-} f(x)$

$m = -2 < 0$

$y = -2x - 3$

$(0, -3)$

$$f(x) = \begin{cases} -2x - 3, & x \le 0 \\ x^2 - 3, & 0 < x \le 2 \end{cases}$$

$y = x^2 - 3$ on $(0, 2]$

$$f(x) = \begin{cases} \dots \\ x^2 - 3, & 0 < x \le 2 \\ 3x - 5, & 2 < x \end{cases}$$

$y = x^2 - 3$ on $(0, 2]$

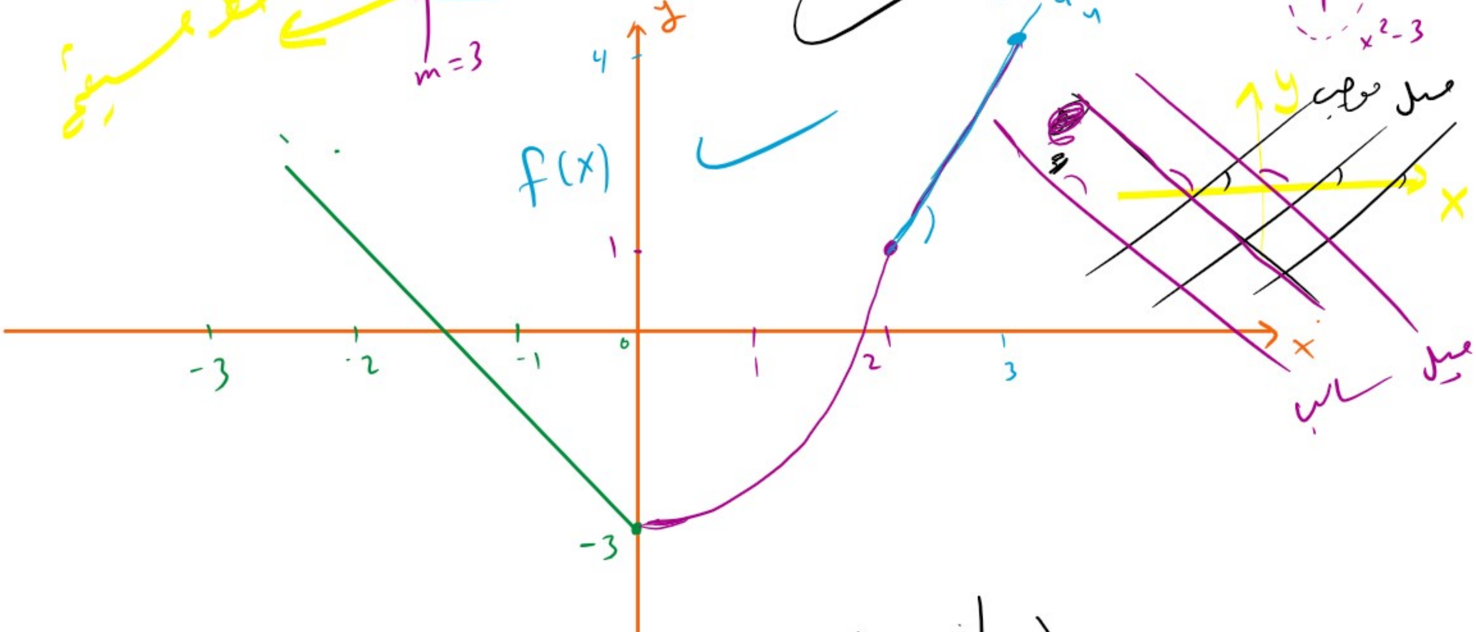
$2^2 - 3 = 4 - 3 = 1$

قطع مستقيم

$m = 3$

$y = 3x - 5$

$9 - 5 = 4$



Exp (Removable Discontinuity)

$f(1) = \frac{0}{0}$

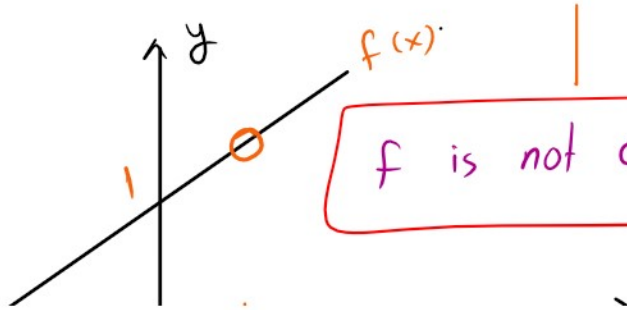
$\Rightarrow f(x) = \dfrac{x^2 - 1}{x - 1} \Rightarrow D(f) = \mathbb{R} \setminus \{1\}$

1) Find $\lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1} = \lim\limits_{x \to 1} \dfrac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}$ $\quad \frac{0}{0}$

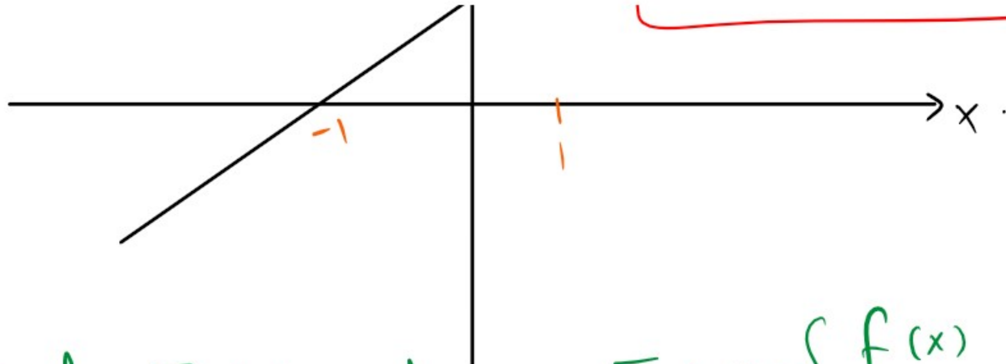
$= \lim\limits_{x \to 1} (x + 1) = (1) + 1 = 2$

2) Sketch $f(x)$

$\boxed{x+1}$



$\boxed{f \text{ is not cont. at } x = 1}$



3) Find $F(x)$ where $F(x) = \begin{cases} f(x) & \text{if } x \neq 1 \\ \lim_{x \to 1} f(x) & \text{if } x = 1 \end{cases}$

$= \begin{cases} \frac{x^2 - 1}{x - 1} & \text{if } x \neq 1 \\ 2 & \text{if } \underline{x = 1} \end{cases}$

Continuous Extension for $f(x)$

$\Rightarrow$ $\boxed{F(x) \text{ is cont. at } x = 1}$

$\underline{F(1)} = \lim_{x \to 1} F(x)$

$2 = 2$

4) What is the difference between $f(x)$ and $F(x)$

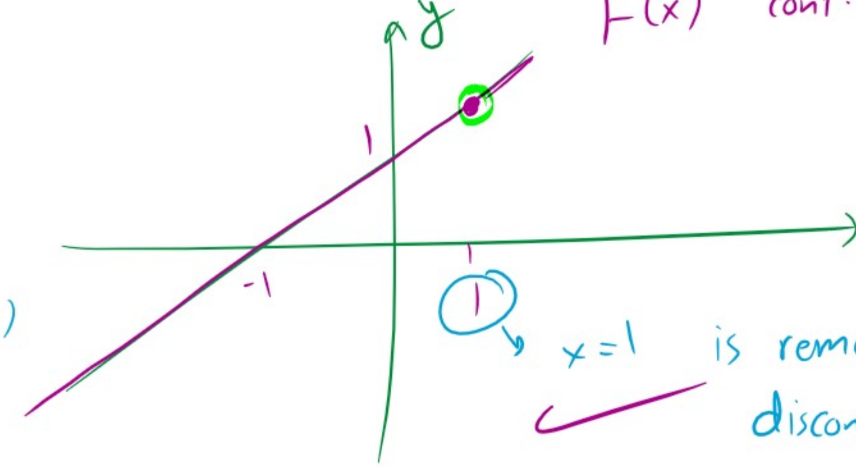
$F(x)$ is cont. at $x = 1$

$f(x)$ is not cont. at $x = 1$

5) sketch $\underline{F(x)}$

Continuous Extension for $f(x)$ at $x = 1$

$F(x)$ cont. at 1



$x = 1$ is removable discontinu

at $x=1$ discontinu

$f(2)=\frac{0}{0}$

Exp Find cont. extension for $f(x)=\dfrac{x^2-5x+6}{2x-4}$ at $x=2$

$D(f)=\mathbb{R}\setminus\{2\}$

f is not cont. at $x=2$

$$\lim_{x\to 2} f(x)=\lim_{x\to 2}\frac{x^2-5x+6}{2x-4}$$

$$=\lim_{x\to 2}\frac{\cancel{(x-2)}(x-3)}{2\cancel{(x-2)}}$$

$$=\lim_{x\to 2}\frac{(x-3)}{2}$$

$$=\frac{\boxed{2}-3}{2}$$

$$=-\frac{1}{2}$$

$$F(x)=\begin{cases} f(x) & \text{if } x\neq 2 \\ \lim\limits_{x\to 2} f(x) & \text{if } x=2 \end{cases}$$

$$F(x)=\begin{cases} \dfrac{x^2-5x+6}{2x-4} & \text{if } x\neq 2 \\ \boxed{-\frac{1}{2}} & \text{if } x=2 \end{cases}$$

$$F(2)=-\frac{1}{2}=\lim_{x\to 2}F(x)=\lim_{x\to 2}\frac{x^2-5x+6}{2x-4}$$

$F(x)$ cont. at $x=2$

$x=2\notin D(f)$ → removable discont.

$f(x)=\dfrac{\square}{\square}$, $x=c$ → removable if $f(c)=\frac{0}{0}$

removable if $f(c) = \frac{0}{0}$

if $f(c) \neq \frac{\text{رقم غير 0}}{0} \Rightarrow c$ is not removable discont.

(c is vertical Asymptote)

V. Asy

Remark: If c is removable $\Rightarrow$ c is not V. Asy

If c is V. Asy $\Rightarrow$ c is not removable

Exp Is $x=0$ removable discont. for $f(x) = \frac{1}{x}$

No $x=0$ is not removable since $f(0) = \frac{1}{0} \neq \frac{0}{0}$

$\Downarrow$

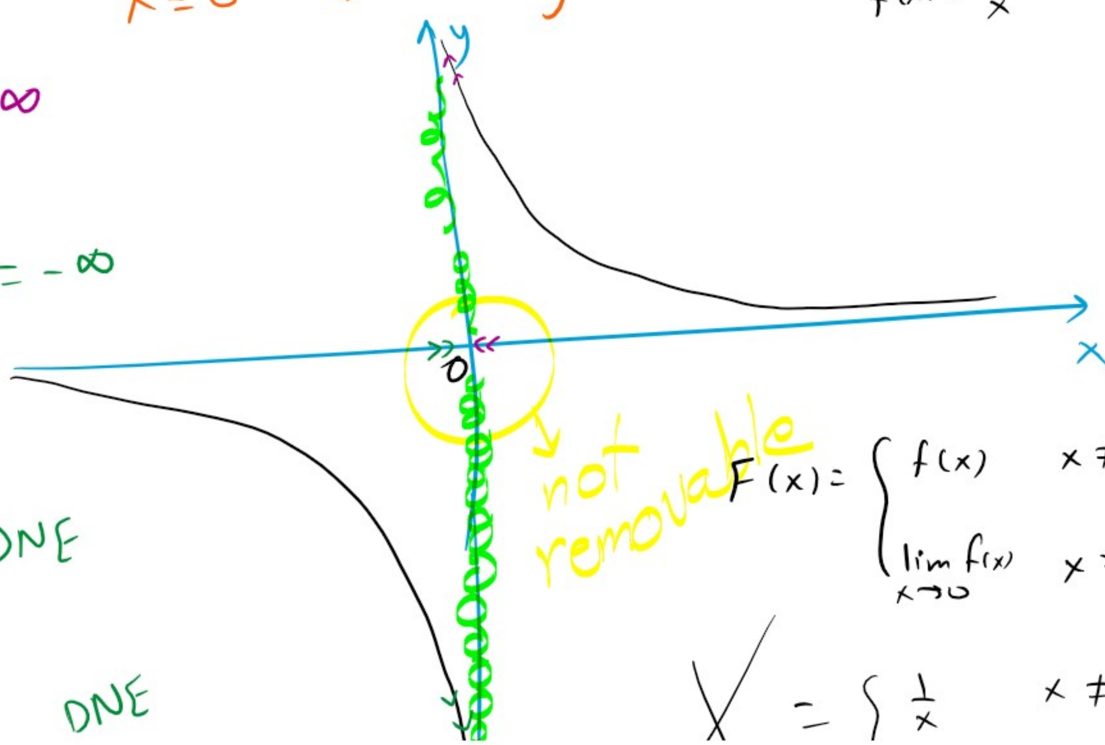
$x=0$ is V. Asy.

$f(x) = \frac{1}{x}$

① $\lim_{x \to 0^+} f(x) = \infty$

② $\lim_{x \to 0^-} f(x) = -\infty$

$\lim_{x \to 0} f(x)$ DNE



$F(x) = \begin{cases} f(x) & x \neq 0 \\ \lim_{x \to 0} f(x) & x = 0 \end{cases}$

$\cancel{F} = \begin{cases} \frac{1}{x} & x \neq 0 \end{cases}$

L L DNE

$\lim_{x \to 0} \frac{1}{x}$ DNE

$\boxed{x=0}$

V. Asy.

$X = \begin{cases} \frac{1}{x} & x \neq 0 \\ \text{DNE} & x = 0 \end{cases}$

Exp $f(x) = \frac{x}{x-1}$

① $\lim_{x \to 0} f(x) = \frac{0}{0-1} = \frac{0}{-1} = 0$

f cont. at $x=0$

$D(f) = \mathbb{R} \setminus \{1\}$

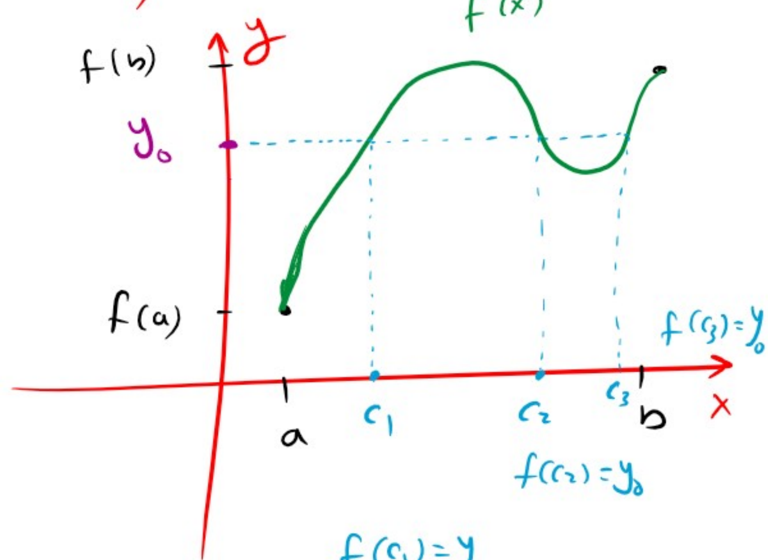
Is $x=1$ removable? $\Rightarrow f(1) = \frac{1}{0}$

NO $\Rightarrow$ $\boxed{x=1 \text{ is V. Asy.}}$

Th (Intermediate Value Th) - IVT

- f cont on $[a,b]$
- $y_0 \in (f(a), f(b))$

Then $\exists$ at least number $c \in [a,b]$ s.t $f(c) = y_0$



حالة خاصة من IVT

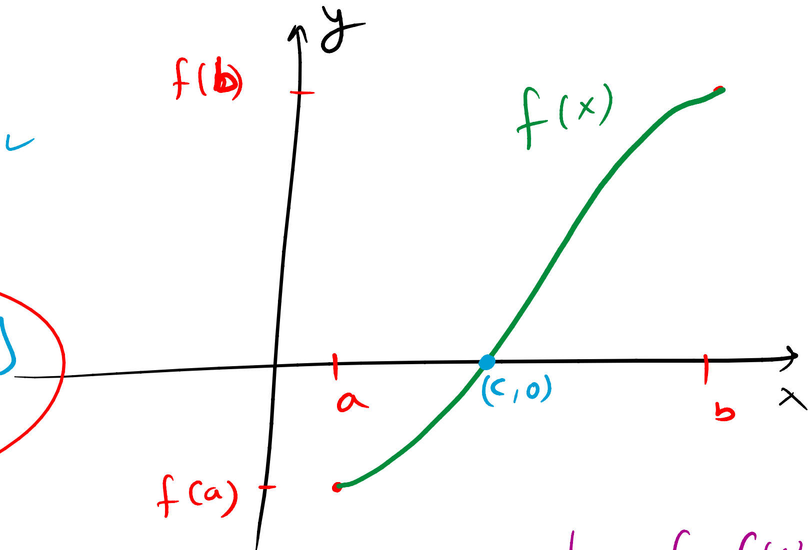
Th (Bolzano Th)

عندما تكون $y_0 = 0$

- f cont. on $[a, b]$ ✓
- $f(a)\, f(b) < 0$ ✓

Then $\exists$ number $c \in [a, b]$

s.t $f(c) = 0 \rightarrow y_0$

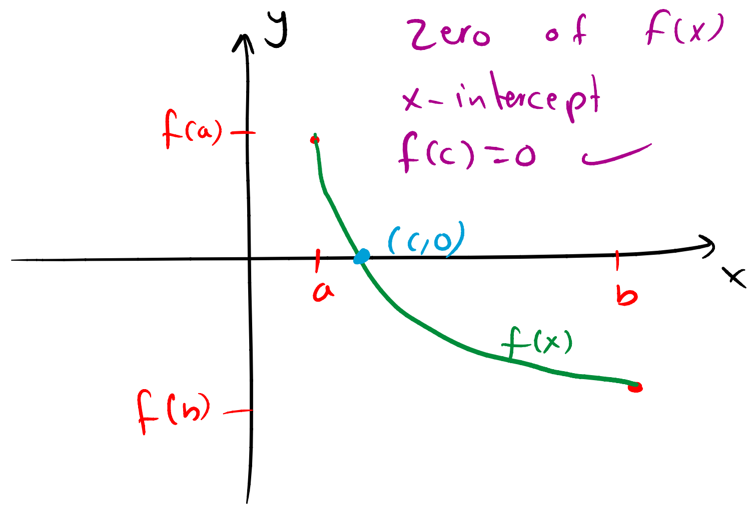


c root of f(x)

zero of f(x)

x-intercept

$f(c) = 0$ ✓



Exp Is there any root for the eq.

$$x^5 - 2x^3 - 2 = 0 \quad ??$$

$f(x) = 0 \quad \Rightarrow \quad f(x) = x^5 - 2x^3 - 2$ cont. on $\mathbb{R}$

كثير حدود

Polynomial

$[a, b] = [0, 2]$

$[a,b] = [0, 2]$

$f(a)<0$ $\quad$ $f(b)>0$

$f(0) = 0-0-2 < 0$

$f(1) = 1^5 - 2 - 2 = 1-2-2 < 0$ $\times$

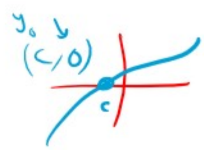
$f(2) = 2^5 - 2(2)^3 - 2$

$= 32 - 16 - 2 > 0$

- f cont on $[0, 2]$
- $f(0)\, f(2) < 0$

By Bolzano $\Rightarrow$ $\exists$ a number $c \in (0, 2)$ s.t $f(c) = 0$

$\exists$ root c



Exp $f(x) = x^2 - 9$ on $[0, 4]$

Does f intersect x-axis in $[0, 4]$

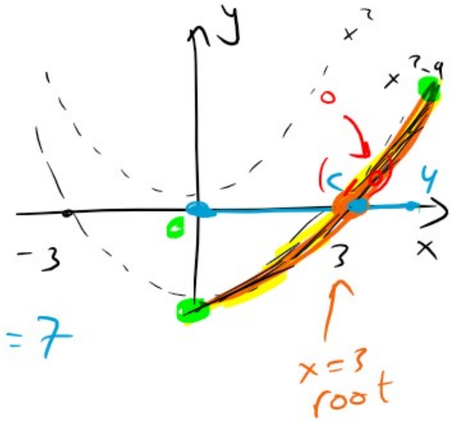
Bolzano

- f cont. on $[0, 4]$ poly.
- $f(0)\, f(4) \overset{??}{<} 0$

$(-9)(7) < 0$ ✓

$f(0) = 0^2 - 9 = -9$

$f(4) = 4^2 - 9 = 16 - 9 = 7$



By Bolzano $\Rightarrow \exists\ c \in (0,4)$ s.t $f(c)=0$

$c=3$

$f(3)=0$

Apply **IVT** $\quad f(x) = x^2 - 9 \quad , [0,4]$

- f cont on $[0,4]$
- $y_0 = 0 \overset{??}{\in} (f(0), f(4))$ $\Rightarrow y_0 = 0 \in (-9, 7)$

$$f(0) = -9, \quad f(4) = 7$$

Then $\exists\ c \in (a,b)$ s.t $f(c) = y_0$

$$\boxed{f(c) = 0}$$

c root