Chap 8: Techniques of integration

8.1:- Integrals by Parts

$$\int u\,dv = uv - \int v\,du$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

- How to solve Exercises?

- you need to decide which of the two functions is u and the other will represent dv
- u doesn't have to be the first function
- After that u need to organize the functions let's say we have $\int x \cos x \, dx$

$\rightarrow u = x \qquad dv = \cos x \, dx$

نشتق $\quad du = dx \qquad v = \sin x \quad$ نكامل

$u \times v$

$-\int v\,du$

- There is another way to solve These integrals Note: it works only if one of the functions can be derived to Zero.

Ex: $x \underset{\text{نشتق}}{\Longrightarrow} 1 \underset{\text{نشتق}}{\Longrightarrow} 0$ ✓

if so :-

d	$\int$
x	$\cos x$
$\downarrow$	$\downarrow$
1	$\sin x$
$\downarrow$	$\downarrow$
0	$-\cos x$

(+)

(−)

some integrals that you have to Know By heart :-

* $\int \ln x \, dx$

$u = \ln x$ $\qquad dv = dx$

$du = \frac{1}{x}$ $\qquad v = x$

$$\int \ln x \, dx = \int u \, dv$$

$$= x \ln x - \int \frac{1}{\cancel{x}} (\cancel{x}) \, dx$$

$$= x \ln x - x + C$$

* $\int e^x \sin x \, dx$

$u = e^x$ $\qquad dv = \sin x \, dx$

$du = e^x$ $\qquad v = -\cos x$

$$\int e^x \sin x \, dx = \int u \, dv$$

$$= -e^x \cos x + \boxed{\int e^x \cos x \, dx}$$

$\int e^x \cos x \, dx$

$u = e^x \qquad dv = \cos x \, dx$

$du = e^x \qquad v = \sin x$

$\int e^x \cos x \, dx = \int u \, dv$

$= e^x \sin x - \int e^x \sin x$

• Now Back to the equation

$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x$

$2\int e^x \sin x \, dx = e^x (\sin x - \cos x)$

$\int e^x \sin x \, dx = \frac{e^x (\sin x - \cos x)}{2} + c$

طريقة أخرى :-

$e^x \; (x) \; \sin x$

$e^x \; (y) \; -\cos x$

$e^x \; \text{stop} \; -\sin x$

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x + \int e^x (-\sin x) \, dx$$

$$\Rightarrow \int e^x \sin x \, dx = e^x [\sin x - \cos x] - \int e^x \sin x \, dx$$

$$\int e^x \sin x \, dx = \frac{e^x [\sin x - \cos x]}{2}$$

8.2 Trigonometric integrals

⚠ How to integral $\int \cos^m x \sin^n x \, dx$?

Answer :- [1] If m is odd (فردي)

Then $m = 2K+1$

$$\Rightarrow \cos^m x = \cos^{2K+1} x$$
$$= [\cos^2 x]^K \cos x$$
$$= [1-\sin^2 x]^K \cos x$$

Then let $u = \sin x$

and $du = \cos x \, dx$

[2] If n is odd $\Rightarrow n = 2K+1$

$$\sin^n x = \sin^{2K+1} x$$
$$= [\sin^2 x]^K \sin x$$
$$= [1-\cos^2 x]^K \sin x$$

Then let $u = \cos x$

$du = -\sin x \, dx$

[3] If n and m are both even then :-

we use : $\sin^2 x = \frac{1-\cos 2x}{2} \rightarrow \cos 2x = 1-2\sin^2 x$

$\cos^2 x = \frac{1+\cos 2x}{2} \rightarrow \cos 2x = 2\cos^2 x - 1$

• Another Case is when n and m are not powers But constants to the variable x :-

1- $\int \sin mx \cos nx \, dx$

$= \frac{1}{2}\int [\sin(m-n)x + \sin(m+n)x] \, dx$

2- $\int \sin mx \sin nx \, dx$

$= \frac{1}{2}\int [\cos(m-n) - \cos(m+n)x] \, dx$

Identical But instead of − we use +

3- $\int \cos mx \cos nx \, dx$

$= \frac{1}{2}\int [\cos(m-n)x + \cos(m+n)x] \, dx$

• ملاحظة: دائماً تكامل sin ناقص تكامل ال cos

Powers of tan and sec :-

① $\int \tan^3 x \, dx$

$= \int \tan^2 x \tan x \, dx$

$= \int (\sec^2 x - 1)(\tan x) \, dx$ we separate them

$= \int \sec^2 x \tan x \, dx - \int \tan x \, dx$ let $u = \tan x$, $du = \sec^2 x \, dx$

$= \int u \, du - \int \frac{\sin x}{\cos x} dx$

$= \frac{u^2}{2} + \int \frac{(\cos x)'}{\cos x} dx$

$= \frac{\tan^2 x}{2} + \ln|\cos x| + C$ ※

2- $\int \sec^4 x \, dx$

$= \int \sec^2 x \sec^2 x \, dx$

$= \int (1 + \tan^2 x) \sec^2 x \, dx$ let $u = \tan x$

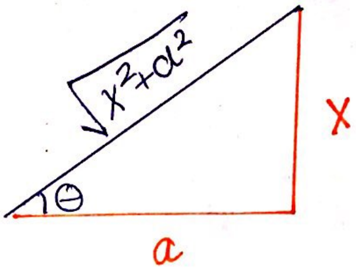
$du = \sec^2 x$

$= \int (1 + u^2) \, du$

$= u + \frac{u^3}{3} + C = \tan x + \frac{\tan^3 x}{3} + C$

Trigonometric substitution

$X = a\tan\theta$

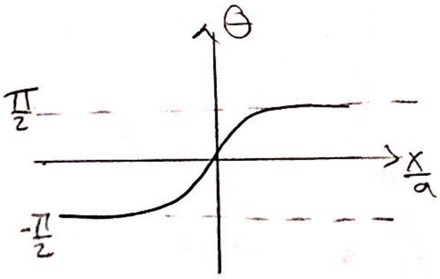


$dX = a\sec^2\theta\, d\theta$

$\theta = \tan^{-1}\left(\frac{X}{a}\right)$

$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

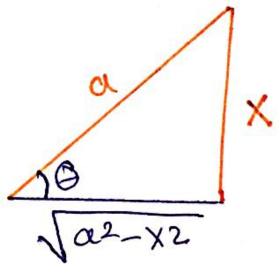
which means $\cos\theta$ is always positive



$\sqrt{X^2+a^2} = \sqrt{a^2\tan^2\theta + a^2} = \sqrt{a^2(\tan^2\theta+1)} = \sqrt{a^2\sec^2\theta} = a\,|\sec\theta|$

But $\sec\theta > 0$ Then $= a\sec\theta$

$X = a\sin\theta$

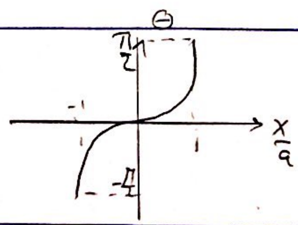


$d\theta = a\cos\theta\, d\theta$

$\theta = \sin^{-1}\left(\frac{X}{a}\right)$

$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

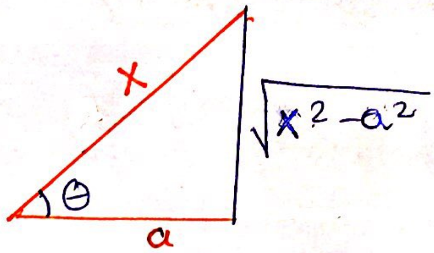
which means $\cos\theta$ is always positive



$\sqrt{a^2-x^2} = \sqrt{a^2 - a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\,|\cos\theta|$

But $\cos\theta > 0 \Rightarrow = a\cos\theta$

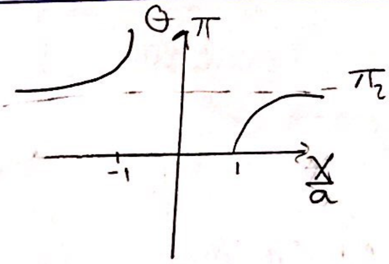
$X = a\sec\Theta$



$dX = a\sec\Theta\tan\Theta \, d\Theta$

$\Theta = \sec^{-1} = \left(\frac{X}{a}\right)$

• There is 2 cases



Case①:- $\frac{X}{a} \geq 1 \Rightarrow 0 \leq \Theta < \frac{\pi}{2}$

case②:- $\frac{X}{a} \leq -1 \Rightarrow \frac{\pi}{2} < \Theta \leq \pi$

In our Book we only deal with Case① :-

$\sqrt{X^2-a^2} = \sqrt{a^2\sec^2\Theta - a^2} = \sqrt{a^2(\sec^2\Theta - 1)} = \sqrt{a^2\tan^2\Theta} = a|\tan\Theta|$

since we are only dealing with Case ①

Then $\Rightarrow = a\tan\Theta$

4 Integration of Rational functions using Partial fractions

1

* A Rational function is : $\frac{f(x)}{g(x)}$

• How to integrate it?

→ The idea of using Partial fraction is to re-write the Rational function $\frac{f(x)}{g(x)}$ as a sum of Partial "simpler" fractions That are easy to integrate

• There are 2 cases:-

[1] The degree of $f(x) \geq$ The degree of $g(x)$

→ we use long division

[2] The degree of $f(x) <$ ~~g~~ The degree of $g(x)$

→ we have the following cases:-

[1a] if $g(x)$ is a product of linear distinct factors

The we use $\boxed{\text{Cover Method}}$

[1b] other-wise we use different approaches

• Remark:- If in case a The products were not distinct : for example:-

$\frac{1}{(x-1)^2}$ we use the cover method but with some changes

$$\frac{1}{(x-1)^2} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2}$$

Another example:-

$$\frac{1}{(X^2+1)^2} = \frac{Ax+B}{X^2+1} + \frac{Cx+1}{(X^2+1)^2}$$

Explanation of the cover method:-

for example: $\int \frac{(X+5)}{(X-4)(X+1)}$

$$\frac{(X+5)}{(X-4)(X+1)} = \frac{A}{(X-4)} + \frac{B}{(X+1)}$$

$$\frac{(X+5)}{(X-4)(X+1)} = \frac{A(X+1) + B(X-4)}{(X-4)(X+1)}$$ بعد توحيد المقامات

$$X+5 = A(X+1) + B(X-4)$$

الأن نختار اي قيمة عشوائية
لـ X لكي نجد A و B
ونفضل ان تكون القيم
تصفر معامل A مرة وتصفر
معامل B مرة اخرى
وهنا تكون هذه القيم هي
$X = -1 / 4$

① $-1+5 = 0 + -5B$

$B = \frac{-4}{5}$

② $4+5 = 5A + 0$

$A = \cancel{\frac{5}{4}}\ \frac{9}{5}$

or :-

$$\frac{(X+5)}{(X-4)(X+1)} = \frac{A}{(X-4)} + \frac{B}{(X+1)}$$

لايجاد A
① : أخذ قيمة X
التي تصفر المقام

② نغطي (X-4)
ونعوض
الـ 4
في المعادلة

③ نحصل على قيمة A

$$A = \frac{4+5}{4+1} = \frac{9}{5}$$

④ نجد B بالتعويض x=-1

Explanation of the long division

Exp $\displaystyle\int \frac{x^3}{x^2-2x+1}\,dx$

$$
\begin{array}{r|l}
 & x+2 \\
\hline
x^2-2x+1 & \cancel{x^3} \\
 & -(\cancel{x^3}-2x^2+x) \\
\hline
 & 2\cancel{x^2}-x \\
 & -(2\cancel{x^2}-4x+2) \\
\hline
 & 3x-2
\end{array}
$$

① نضرب المقسوم عليه ب x

② نطرح x^3-2x^2+x من x^3

③ نضرب المقسوم عليه ب 2

④ نطرح $2x^2-4x+2$ من $2x^2-x$

⑤ نضع الجواب + الباقي

$$\int \frac{x^3}{x^2-2x+1}\,dx = \int (x+2) + \frac{(3x-2)}{(x^2-2x+1)}$$