$\boxed{1.4}$ Matrix Algebra

Th For any scalars α, β and for any matrices A, B, C:

$\boxed{1}$ $A + B = B + A$

$\boxed{2}$ $(A+B)+C = A + (B+C)$

$\boxed{3}$ $(AB)C = A(BC)$

$\boxed{4}$ $A(B+C) = AB + AC$

$\boxed{5}$ $(A+B)C = AC + BC$

$\boxed{6}$ $(\alpha B)A = \alpha(BA)$

$\boxed{7}$ $\alpha(AB) = (\alpha A)B = A(\alpha B)$

$\boxed{8}$ $(\alpha + \beta)A = \alpha A + \beta A$

$\boxed{9}$ $\alpha(A+B) = \alpha A + \alpha B$

Exp Let $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 \\ -3 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$

$\Rightarrow \boxed{3}$ $(AB)C = \left(\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -3 & 4 \end{bmatrix}\right)\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 8 \\ 3 & -4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ -4 & -5 \end{bmatrix}$

$A(BC) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} 2 & 0 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 12 \\ -4 & -5 \end{bmatrix}$

$\Rightarrow \boxed{4}$ $A(B+C) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\left(\begin{bmatrix} 2 & 0 \\ -3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -2 & 6 \end{bmatrix} = \begin{bmatrix} -2 & 13 \\ 2 & -6 \end{bmatrix}$

$AB + AC = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ -3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 8 \\ 3 & -4 \end{bmatrix} + \begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 13 \\ 2 & -6 \end{bmatrix}$

Proof $\boxed{4}$ Let $A = (a_{ij})$ be $m \times n$ matrix and $B = (b_{ij})$, $C = (c_{ij})$ be $n \times r$ matrices.

~~Hence,~~

Let $D = A(B+C)$ and $E = AB + AC$.

$$d_{ij} = \sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) = \sum_{k=1}^{n} a_{ik} b_{kj} + \sum_{k=1}^{n} a_{ik} c_{kj}$$

$$e_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj} + \sum_{k=1}^{n} a_{ik} c_{kj} = d_{ij}$$

Hence, $A(B+C) = AB + AC$.

Notation: If K is positive integer, then

$$A^K = \underbrace{A\,A \cdots A}_{K \text{ times}}$$

Exp If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then $A^2 = AA = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$

$$A^3 = AAA = AA^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

in general $\Rightarrow$ $A^n = \begin{bmatrix} 2^{n-1} & 2^{n-1} \\ 2^{n-1} & 2^{n-1} \end{bmatrix}$

* **The Identity Matrix** is $n \times n$ matrix denoted by

$$I = (\delta_{ij}) \text{ where } \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Exp $I_{2\times2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_{3\times3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, ...

Note $\boxed{1}$ If A is $n\times n$ matrix then $AI = IA = A$

Exp $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 & 4 \\ 2 & 0 & 5 \\ 1 & 6 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 4 \\ 2 & 0 & 5 \\ 1 & 6 & -1 \end{bmatrix}$

$$\begin{bmatrix} 3 & -2 & 4 \\ 2 & 0 & 5 \\ 1 & 6 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 4 \\ 2 & 0 & 5 \\ 1 & 6 & -1 \end{bmatrix}$$

$\boxed{2}$ If I is the $n\times n$ identity Matrix, then its columns are the standard vectors used to define the coordinate system in Euclidean n-space $\mathbb{R}^n$.

$$\Rightarrow \quad I = \underbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{n\times n} = (e_1, e_2, \dots, e_n).$$

* Matrix Inversion

* The real number a has a multiplicative inverse if $\exists$ a number b s.t $ab = 1$

Def The matrix $A_{n \times n}$ is nonsingular (invertible) if $\exists$ a matrix $B_{n \times n}$ s.t $AB = BA = I$. B is called multiplicative inverse of A (B is the inverse of A) denoted by $B = A^{-1}$.

$\boxplus$ The matrix can have at most one inverse

- If the matrix A has two inverses B and C, then $AC = BA = I$
- But $B = BI = B(AC) = (BA)C = IC = C$

The $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ has inverse $A^{-1} = \frac{1}{d}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$ if $d \neq 0$, where $d = |A| = a_{11}a_{22} - a_{12}a_{21}$

Exp If $A = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}$ then $A^{-1} = \frac{1}{-2}\begin{bmatrix} -3 & -4 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 2 \\ -\frac{1}{2} & -1 \end{bmatrix}$

$$A A^{-1} = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} \frac{3}{2} & 2 \\ -\frac{1}{2} & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^{-1}A$$

Exp $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$ are inverses of each other.

because $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and

$$\begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Exp* The matrix $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ has no inverse because $d=|A|=0$. So there will be no $B_{2\times 2}$ s.t $AB=\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}b_{11}&b_{12}\\b_{21}&b_{22}\end{bmatrix}=\begin{bmatrix}b_{11}&b_{12}\\0&0\end{bmatrix}\neq\begin{bmatrix}1&0\\0&1\end{bmatrix}$

Def The $n\times n$ matrix is singular if it does not have a multiplicative inverse. Hence $A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ is singular.

Th If A and B are nonsingular $n\times n$ matrices, then AB is nonsingular given by $(AB)^{-1}=B^{-1}A^{-1}$

Proof $(B^{-1}A^{-1})AB = B^{-1}(A^{-1}A)B = B^{-1}B = I$ and

$AB(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$

Note If $A_1, A_2, \cdots, A_K$ are nonsingular $n\times n$ matrices, then $A_1A_2\cdots A_K$ is nonsingular given by

$$(A_1A_2\cdots A_K)^{-1} = A_K^{-1}\cdots A_2^{-1}A_1^{-1}.$$

Exp Let A and B be $n\times n$ matrices. Show that if $AB=A$ and $B\neq I$ then A must be singular.

suppose not, i.e. $AB=A$ and $B\neq I$ and A is non-singular

$\Rightarrow$ A has an inverse A^{-1} s.t $AA^{-1}=A^{-1}A=I$

$A^{-1}AB=A^{-1}A$

$B=I$ $\times$ since $B\neq I$

Hence, A is singular.

Exp Let A be nonsingular matrix. show that
[1] A^{-1} is nonsingular [2] $(A^{-1})^{-1} = A$

[1] since A is nonsingular $\Rightarrow$ $AA^{-1} = A^{-1}A = I$
Take the inverse for both sides $(A^{-1})^{-1}A^{-1} = A^{-1}(A^{-1})^{-1} = I$
Hence, A^{-1} is nonsingular

[2] From part [1] we have $A^{-1}(A^{-1})^{-1} = I$
Multiply both sides by A $\quad (A^{-1})^{-1} = A$

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* Algebraic Rules for Transpose:

[1] $(A^T)^T = A$ $\qquad$ [2] $(\alpha A)^T = \alpha A^T$

[3] $(A+B)^T = A^T + B^T$ $\qquad$ [4] $(AB)^T = B^T A^T$

Exp Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix}$

[4] $AB = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 5 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 6 & 5 \\ 34 & 23 & 14 \\ 15 & \boxed{8} & 9 \end{bmatrix}$

$(AB)^T = \begin{bmatrix} 10 & 34 & 15 \\ 6 & 23 & \boxed{8} \\ 5 & 14 & 9 \end{bmatrix}$

$B^T A^T = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 1 & 4 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 2 & 3 & 4 \\ 1 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 34 & 15 \\ 6 & 23 & \boxed{8} \\ 5 & 14 & 9 \end{bmatrix} = (AB)^T$

Exp show that if A is $m \times n$ matrix then $A^T A$ and AA^T are both symmetric $(A^T A)^T = A^T (A^T)^T = A^T A$ Hence, $A^T A$ is symmetric

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Def The Matrix A is idempotent if $A^2 = A$.

Example $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$