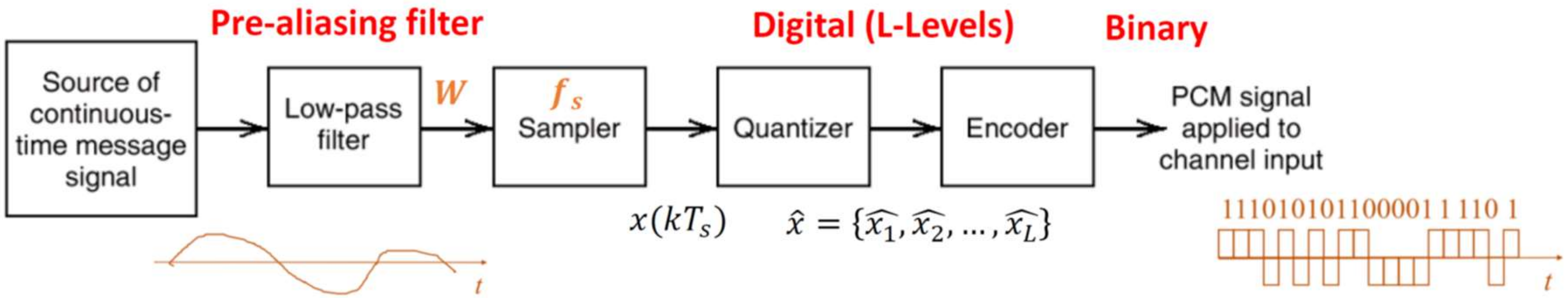
Pulse Modulation

Sampling Theorem
Lecture Outline

- In this this, and the next few lecture, we will address the subject of **pulse code modulation**, where an analog source can be converted into a digital waveform via **sampling**, **quantization**, and **binary encoding**.
- This lecture focuses on Ideal Sampling and the Sampling Theorem.
- The phenomenon of aliasing is explained in detail.
- In the next lecture, we will present two other sampling techniques, which are natural and flat-topped sampling.

Pulse Code Modulation

- Sources are of two types; **analog and digital**. For an analog source, the input transducer is used to convert the physical message generated by the source into a time-varying electrical signal called the *message signal* (like the human voice). This is a continuous time continuous amplitude signal.
- An analog source can be converted into digital via sampling, quantization, and encoding. This process is called **pulse code modulation**



- **Sampler:** If W is the highest frequency component in a signal, then the sampling rate required to reconstruct the message from its samples should follow the Nyquist Rate where $f_s > 2W$.
- The output of the sampler is a continuous amplitude discrete time signal.
- **Quantizer:** Converts the continuous amplitude samples $x(kT_s)$ into **discrete** level samples $\hat{x}(kT_s)$ taken from a finite set of L possible values $\hat{x} = \{\widehat{x_1}, \widehat{x_2}, \dots, \widehat{x_L}\}$.
- **Binary Encoder:** Each quantized level is represented by $r = log_2 L$ binary digits

Sampling Techniques

- **Sampling:** is the process by which a continuous time continuous amplitude signal is converted into a discrete time continuous amplitude signal.

There are three types of sampling:

- Ideal sampling: To be presented in this lecture
- Natural sampling: To be discussed in the next lecture
- Flat-topped sampling (sample and hold): To be discussed in the next lecture.

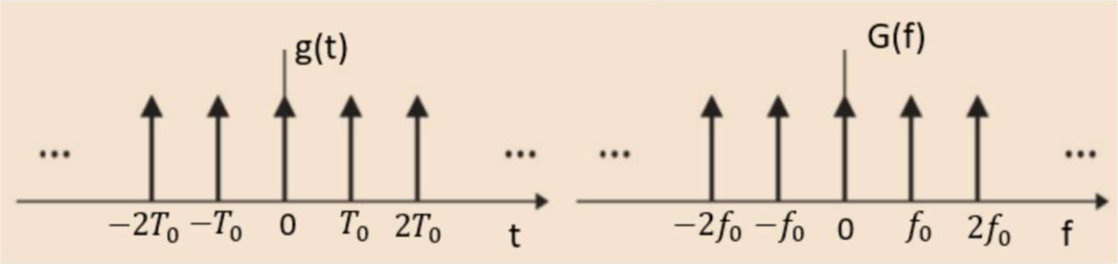
Ideal Sampling: The Periodic Train of Impulses

- **Periodic Signals**: A periodic signal g(t) is expanded in the complex Fourier series form as:
- $g(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \Rightarrow \Im\{g(t) = \sum_{n=-\infty}^{\infty} C_n\, \delta(f - nf_0)$

Example: Consider the following train of impulses $\boldsymbol{g(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0)}$

Solution: The Fourier coefficients are obtained by integrating over one period of $g(t)$.

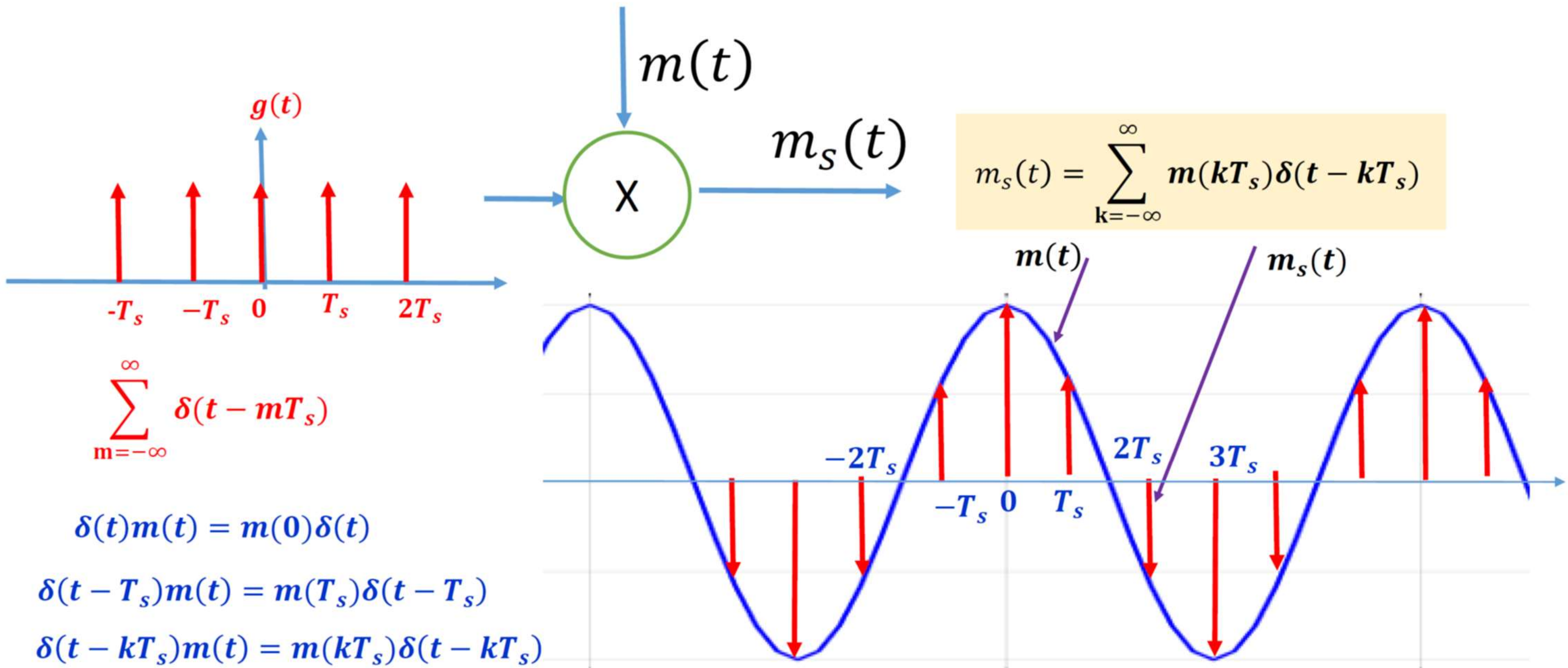
- $C_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} g(t)\, e^{-jn\omega_0 t} dt = \frac{1}{T_0} = f_0$; Note that the sifting property has been used.
- Therefore, the complex Fourier series of $g(t)$ is
- $g(t) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}; \Rightarrow \Im\{g(t)\} = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \Im\{e^{jn\omega_0 t}\} = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f - nf_0)$
- $\boldsymbol{\Im \sum_{m=-\infty}^{\infty} \delta(t - mT_0) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f - nf_0)}$.



Remark 1: Note that the signal is periodic in the time domain and its Fourier transform is periodic in the frequency domain.
Remark 2: This sequence will be found useful when the sampling theorem is considered later in this lecture.

Ideal Sampling

- **Ideal Sampling**: The message m(t), with Fourier transform M(f), which is band-limited to W Hz, is multiplied by a periodic sequence of ideal impulses with period T_s to produce the sampled signal $m_s(t)$.



Ideal Sampling

- $m_s(t) = m(t)g(t) = m(t) \sum_{m=-\infty}^{\infty} \delta(t - mT_s) = \sum_{k=-\infty}^{\infty} m(kT_s)\delta(t - kT_s)$
- Recall the Fourier transform pair: $\boldsymbol{G(f) = \mathfrak{F}(g(t)) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} \delta(f - nf_0)}$
- Hence, $m_s(t) = m(t)\frac{1}{T_0}\sum_{n=-\infty}^{\infty} e^{jn\omega_0 t}$; Product of two functions in the time domain.
- The Fourier transform of $m_s(t)$ is the convolution of M(f) and G(f)

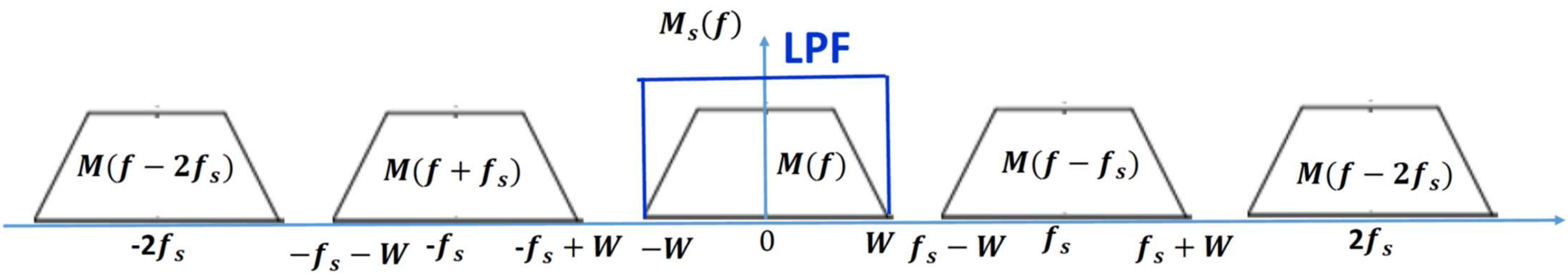
- $M_s(f) = M(f) * G(f) = M(f) * \frac{1}{T_s}\sum_{n=-\infty}^{\infty} \delta(f - nf_s)$

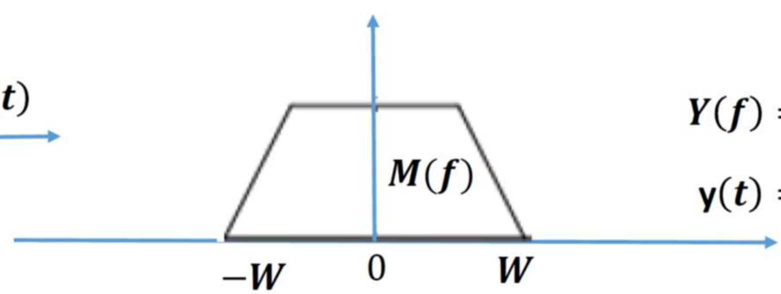
$$\boldsymbol{M_s(f) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} M(f - nf_s)}$$

Ideal Sampling: $f_s > 2W$

$$M_s(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} M(f - nf_s)$$

- Let M(f) be as given in the figure. When $f_s > 2W$, $M_s(f)$ will look like





$$Y(f) = \frac{1}{T_s} M(f)$$

$$y(t) = \frac{1}{T_s} m(t)$$

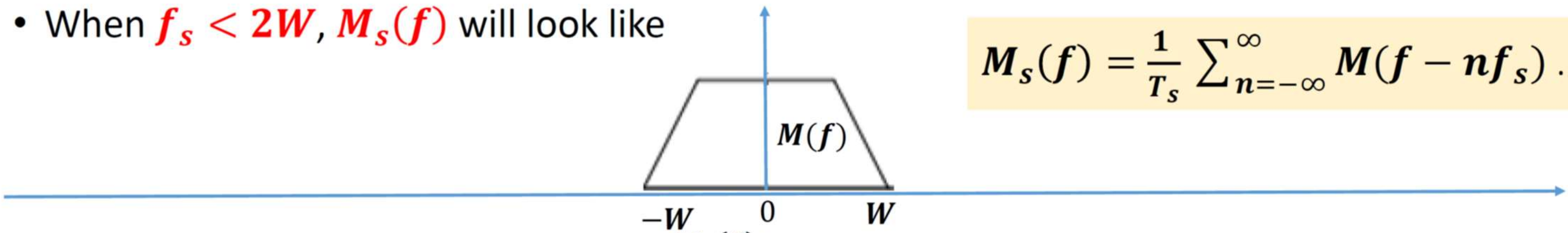
The Sampling Theorem

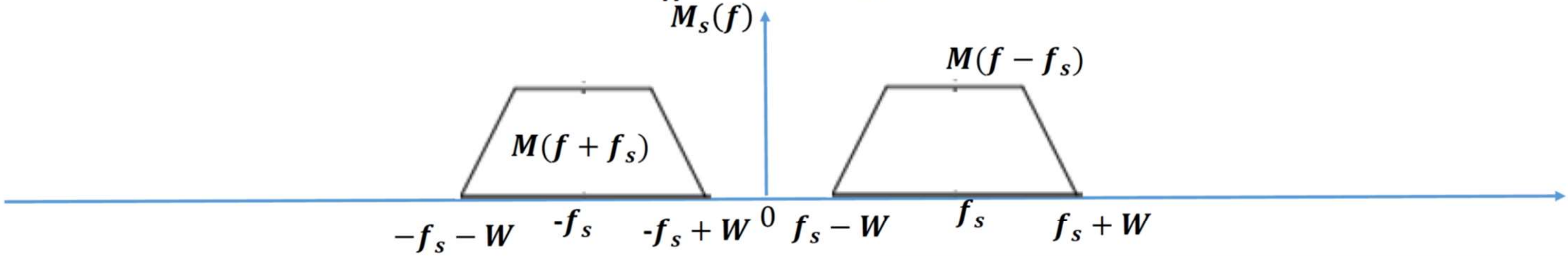
- A bandlimited signal with no frequency components above W Hz can be recovered uniquely from its samples taken every T_s seconds provided that $f_s \geq 2W$, where $f_s = 1/T_s$ is the sampling rate is samples/sec.
- The message m(t) can be recovered from $m_s(t)$ using an ideal LPF with bandwidth W.
- The Sampling frequency $f_s = 2W$, is called the Nyquist rate. It represents the minimum rate at which a signal must be sampled in order to reconstruct it from its samples without distortion.
- When the sampling rate is less than the Nyquist rate, a distortion type of noise called **Aliasing** results.

Sampling Theorem and Aliasing

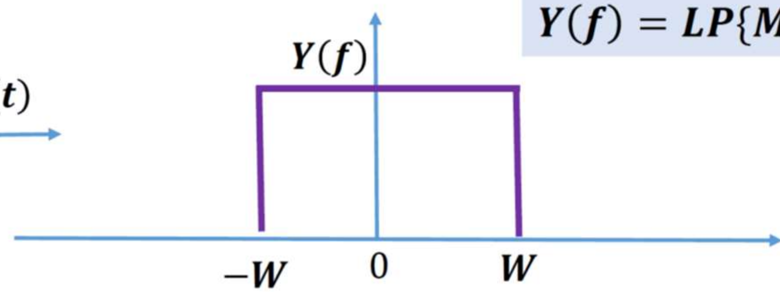
- Let M(f) be the Fourier transform of the message m(t).
- When $f_s < 2W$, $M_s(f)$ will look like

$$M_s(f) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} M(f - nf_s)\,.$$





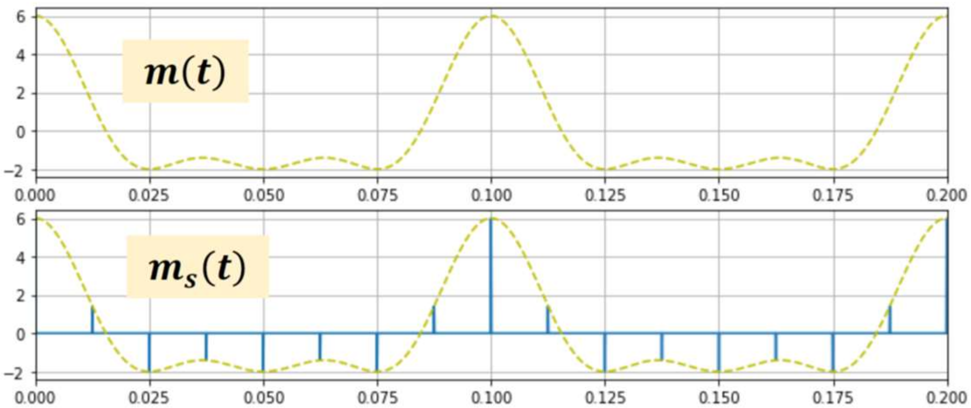
$$Y(f) = LP\{M(f) + M(f - f_s) + M(f + f_s)\}$$



$$Y(f) \neq \frac{1}{T_s} M(f)$$

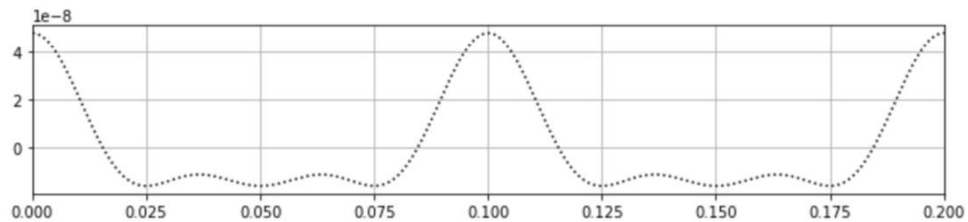
$$y(t) \neq \frac{1}{T_s} m(t)$$

Sampling Theorem: Example $f_s > 2W$

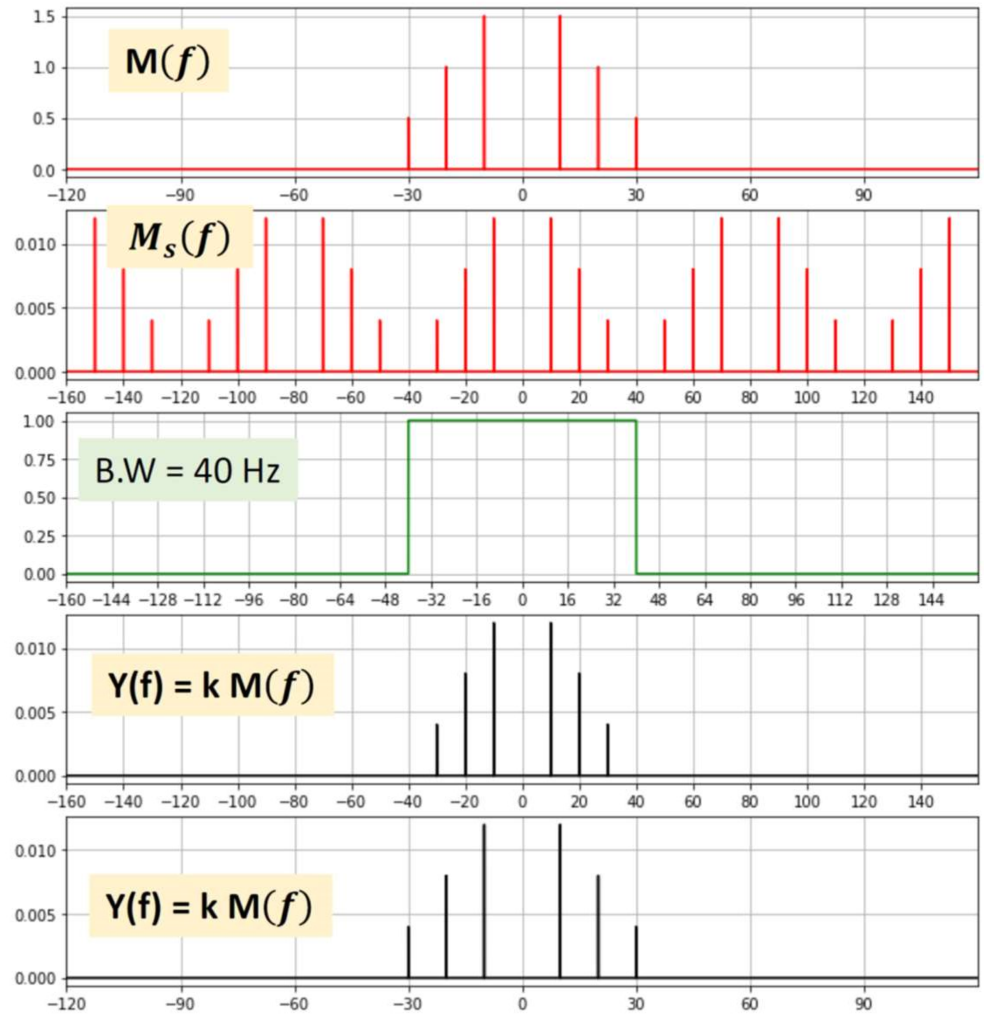


m(t) = 3cos2π(10)t+2cos2π(20)t+cos2π(30)t

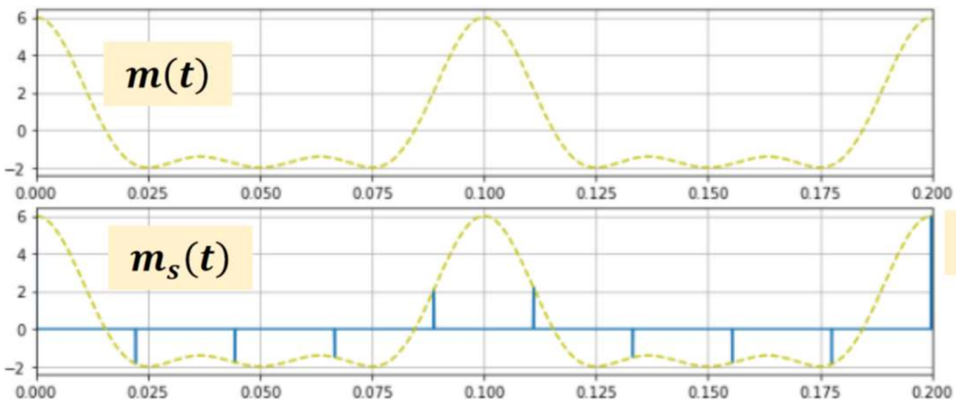
f_s= 80; T_s= 0.0125



- **Since the sampling rate is greater than the Nyquist rate, the original signal is recovered without distortion.**
- **Output contains the message frequencies: 10, 20, 30**

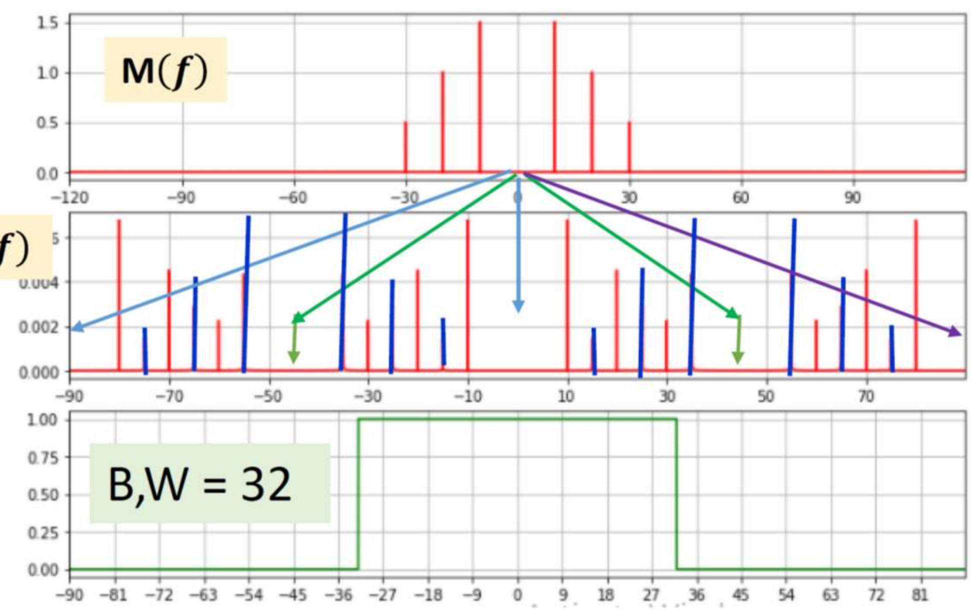


Sampling Theorem and Aliasing : $f_s < 2W$



m(t) = 3cos2π(10)t+2cos2π(20)t+cos2π(30)t

f_s= 45; T_s= 1/45 = 0.022



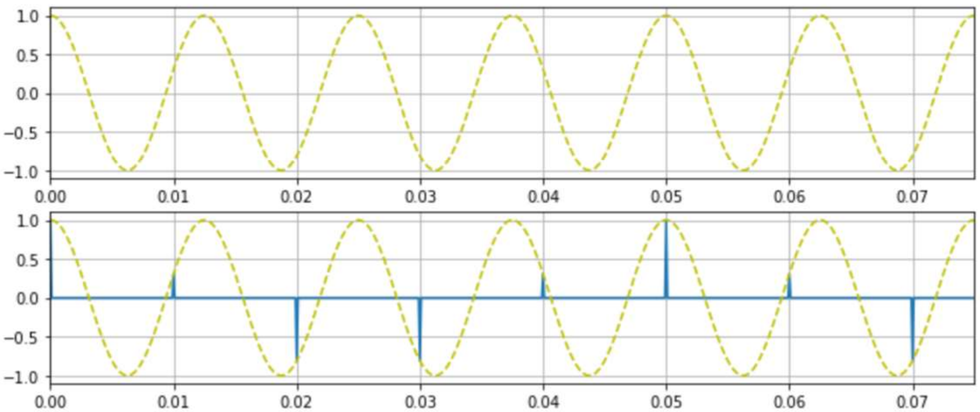
$$M_s(f) = \frac{1}{T_s}\{M(f) + M(f - f_s) + M(f + f_s) + \cdots\}$$

Output contains the message frequencies: 10, 20, 30 Hz. In addition to aliasing frequencies within message bandwidth (45 – 20) = 25 and (45 – 30) = 15

y(t) = k{3cos2π(10)t+2cos2π(20)t+cos2π(30)t

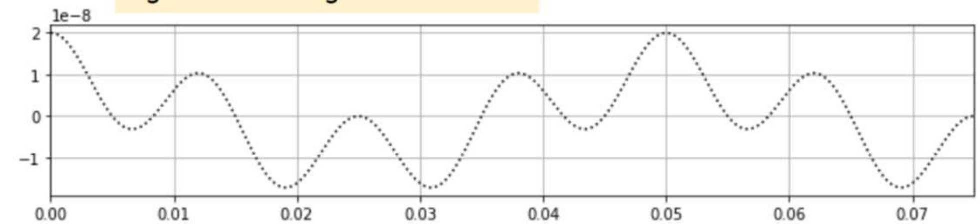
+2cos2π(25)t + cos2π(15)t

Sampling Theorem and Aliasing : Example $f_s < 2W$

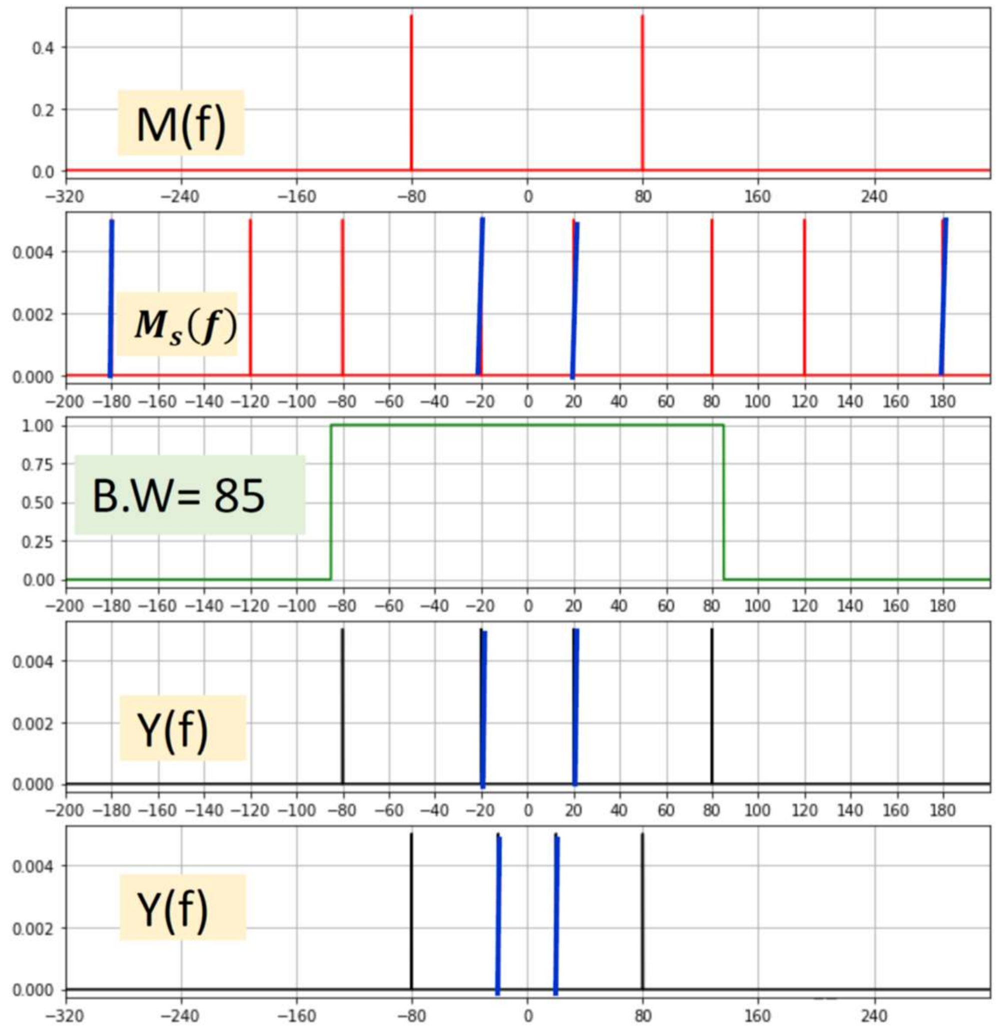


m(t) = cos2π(80)t

f_s= 100; T_s= 0.01 =



y(t) = kcos2π(80)t + kcos2π(20)t



Natural Sampling
Lecture Outline

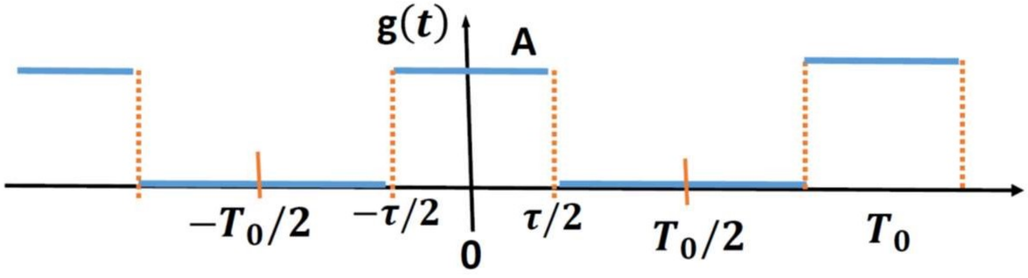
- **Sampling:** is the process by which a continuous time continuous amplitude signal is converted into a discrete time continuous amplitude signal.
- **There are three types of sampling:** Ideal sampling, natural sampling, and flat-topped sampling.
- Ideal sampling, the sampling theorem, and the phenomenon of aliasing were presented in the previous lecture.
- This lecture focuses on natural sampling and the sampling theorem.
- In the next lecture, we consider
  - Flat-topped sampling (sample and hold).
  - Time division multiplexing (TDM)

The Periodic Train of Rectangular Pulses: Fourier Series

- **Example**: Find the trigonometric Fourier series of the periodic rectangular signal defined over one period T_0 as:

$$g(t) = \begin{cases} +A, & -\tau/2 \le t \le \tau/2 \\ 0, & otherwise \end{cases}$$

- **Solution**: The FS is given as $g(t) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$
- $a_0 = \frac{1}{T_0}\int_{-\tau/2}^{\tau/2} g(t)\, dt \Rightarrow \boldsymbol{a_0 = A\tau/T_0}$; $\boldsymbol{\tau/T_0}$ is called the duty cycle of the pulse train.
- $b_n = \frac{2}{T_0}\int_{-\tau/2}^{\tau/2} g(t) \sin(\frac{2\pi n}{T_0} t)\, dt = 0$; g(t) is an even function of t
- $a_n = \frac{2}{T_0}\int_{-\tau/2}^{\tau/2} g(t) \cos(\frac{2\pi n}{T_0} t)\, dt = \frac{4}{T_0}\int_{0}^{\tau/2} A \cos(\frac{2\pi n}{T_0} t)\, dt \Rightarrow \boldsymbol{a_n = \frac{2A}{n\pi} \sin(\frac{n\pi\tau}{T_0})}$
- $a_n = 0$ when $\mathrm{n} = \frac{T_0}{\tau}, \frac{2T_0}{\tau}, \frac{3T_0}{\tau}, \ldots$
- This is demonstrated on the next slide

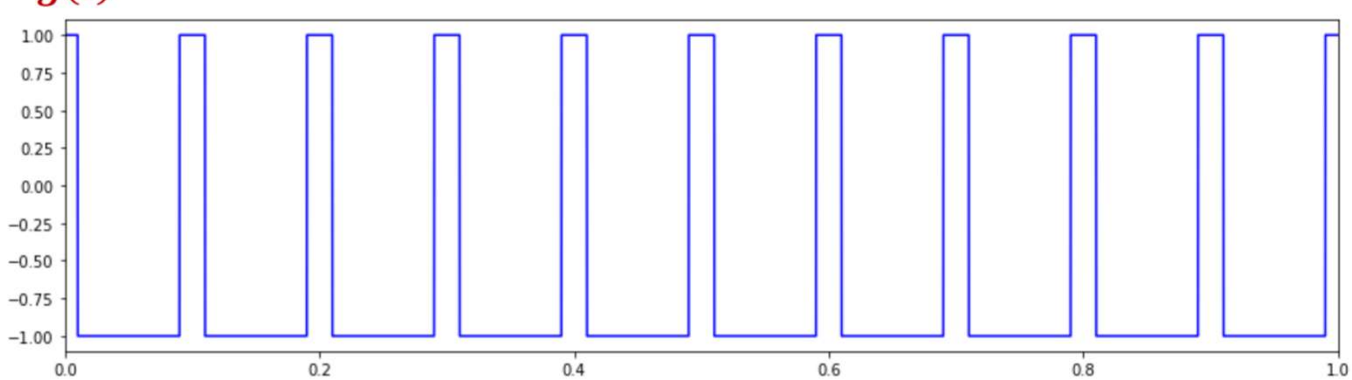


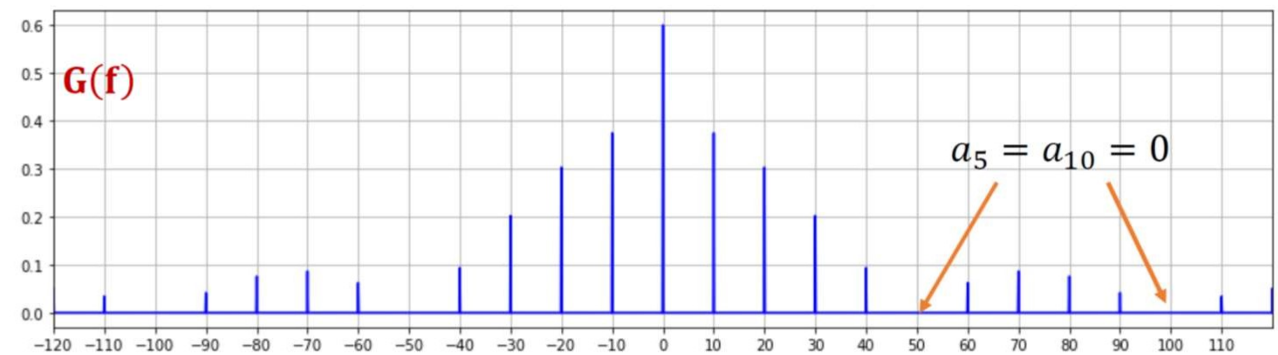
The Periodic Train of Rectangular Pulses: Time and Frequency

The Fourier series of the pulse train is: $g(t) \sim a_0 + \sum_{n=1}^{\infty}(a_n \cos 2\pi n f_0 t)$

The Fourier transform is : $\mathbf{G(f)} = a_0\delta(f) + \sum_{n=1}^{\infty} \frac{a_n}{2}[\delta(f - nf_0) + \delta(f + nf_0)]$

$g(t)$





Example:

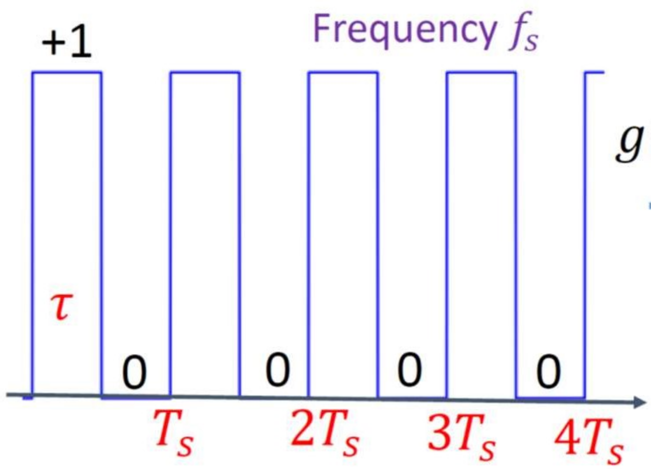
- $f_0 = 10$ Hz; $T_0 = 0.1$
- Duty cycle $\frac{\tau}{T_0} = 0.2$
- $a_n = 0$ when:
  - $n = \frac{T_0}{\tau}, \frac{2T_0}{\tau}, \frac{3T_0}{\tau}, \ldots$
  - $n = 5, 10, 15, \ldots$
- Spectral lines at $5f_0, 10f_0, \ldots$ vanish

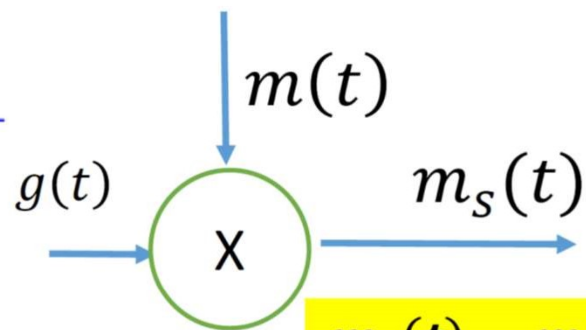
$a_0 = A\tau/T_0$

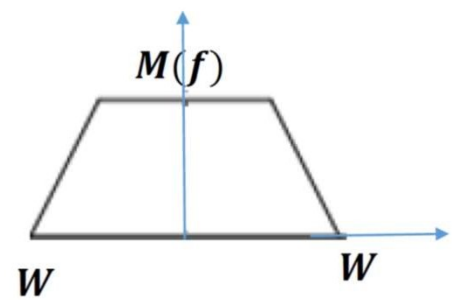
$a_n = \frac{2A}{n\pi}\sin(\frac{n\pi\tau}{T_0})$

Natural Sampling

- **Natural Sampling**: The message m(t), with Fourier transform M(f), which is band-limited to W Hz, is multiplied by a periodic sequence of pulses with period T_s to produce the sampled signal $m_s(t)$.



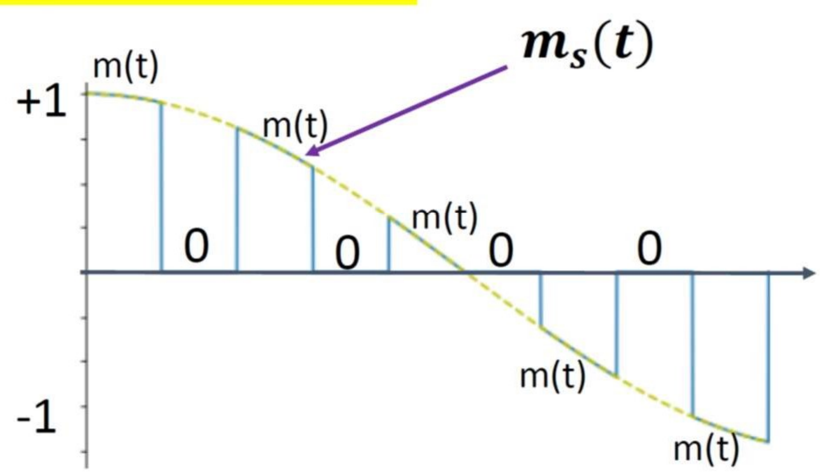




$$\boldsymbol{m_s(t) = m(t)g(t)}$$

$m_s(t)$ = m(t) when g(t) = 1

$m_s(t)$ = 0 when g(t) = 0

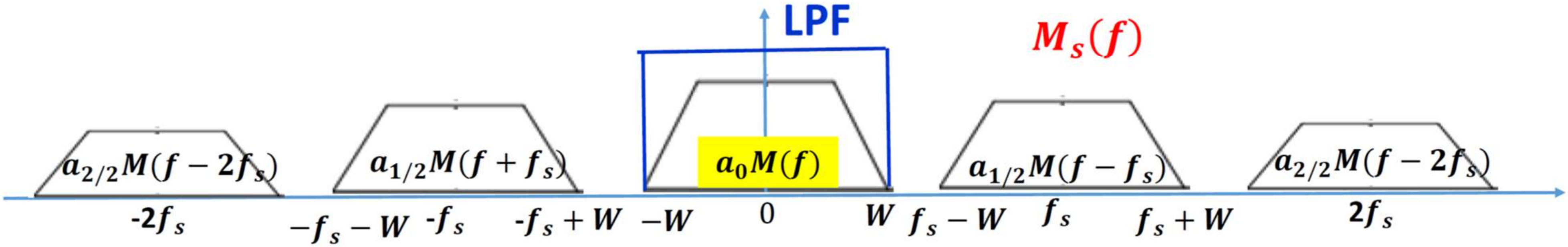


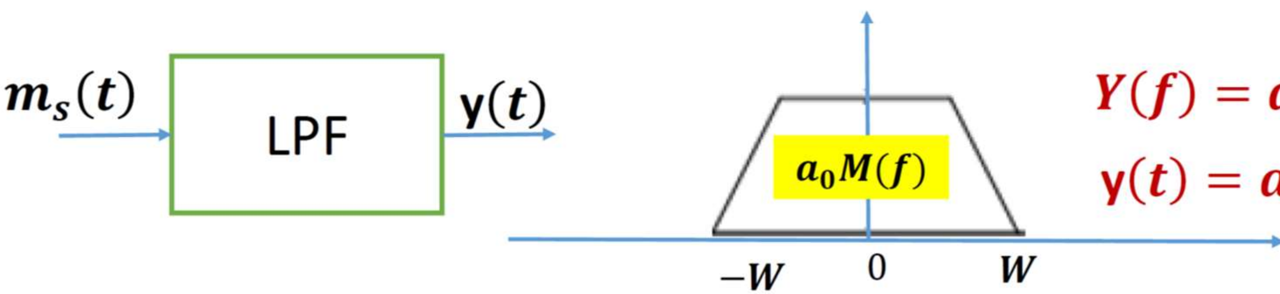
Next, we find M_s(f) in terms of M(f) and $f_s = 1/T_s$

Natural Sampling: $\boldsymbol{f_s > 2W}$

- $m_s(t) = m(t)g(t) = m(t)\{a_0 + \sum_{n=1}^{\infty} a_n \cos 2\pi n f_s t\}$
- $m_s(t) = m(t)g(t) = a_0 m(t) + \sum_{n=1}^{\infty} a_n\, m(t) \cos 2\pi n f_s t$
- $\boldsymbol{M_s(f) = a_0 M(f) + \sum_{n=1}^{\infty} \frac{a_n}{2}[M(f - nf_s) + M(f + nf_s)]}$
- When $\boldsymbol{f_s > 2W}$, $\boldsymbol{M_s(f)}$ will look like

$\boldsymbol{f_s - W \geq W \Rightarrow f_s \geq 2W}$





$\boldsymbol{Y(f) = a_0 M(f)}$

$\boldsymbol{y(t) = a_0 m(t)}$

Message recovered without distortion

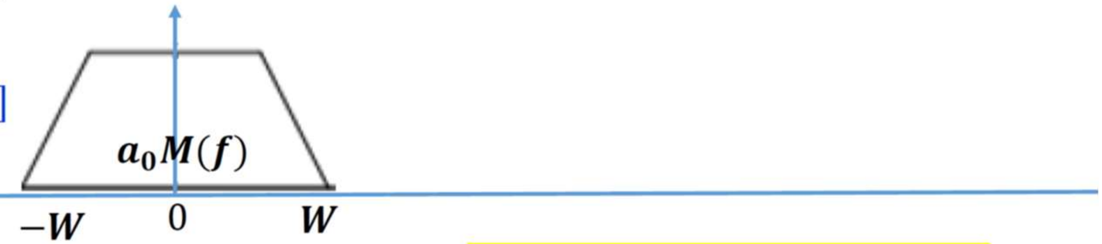
The Sampling Theorem

- A bandlimited signal with no frequency components above W Hz can be recovered uniquely from its samples taken every T_s seconds provided that $f_s \geq 2W$, where $f_s = 1/T_s$ is the sampling rate is samples/sec.
- The message m(t) can be recovered from $m_s(t)$ using an ideal LPF with bandwidth W.
- The Sampling frequency fs = 2W, is called the Nyquist rate. It represents the minimum rate at which a signal must be sampled in order to reconstruct it from its samples without distortion.
- When the sampling rate is less than the Nyquist rate, a distortion type of noise called **Aliasing** results.

Natural Sampling: $f_s < 2W$

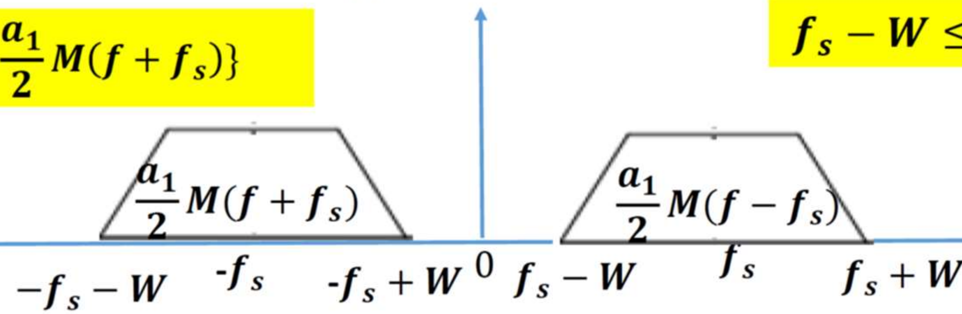
- Let m(t) be the baseband signal with bandwidth W Hz. Assume that $f_s < 2W$
- Need to find $M_s(f)$ and the filtered signal.

$$M_s(f) = a_0 M(f) + \sum_{n=-\infty}^{\infty} \frac{a_n}{2}\left[M(f - nf_s) + M(f + nf_s)\right]$$



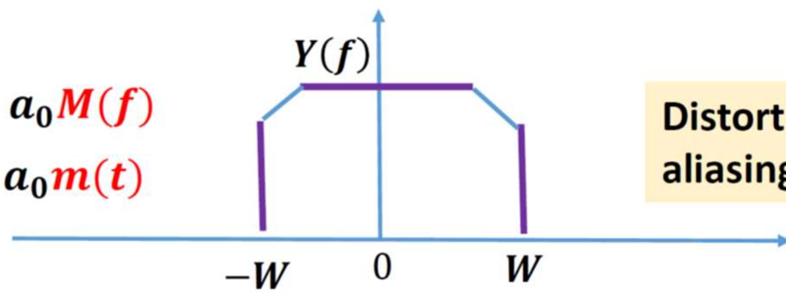
$$Y(f) = LP\{a_0 M(f) + \frac{a_1}{2} M(f - f_s) + \frac{a_1}{2} M(f + f_s)\}$$

$f_s - W \le W \Rightarrow f_s \le 2W$



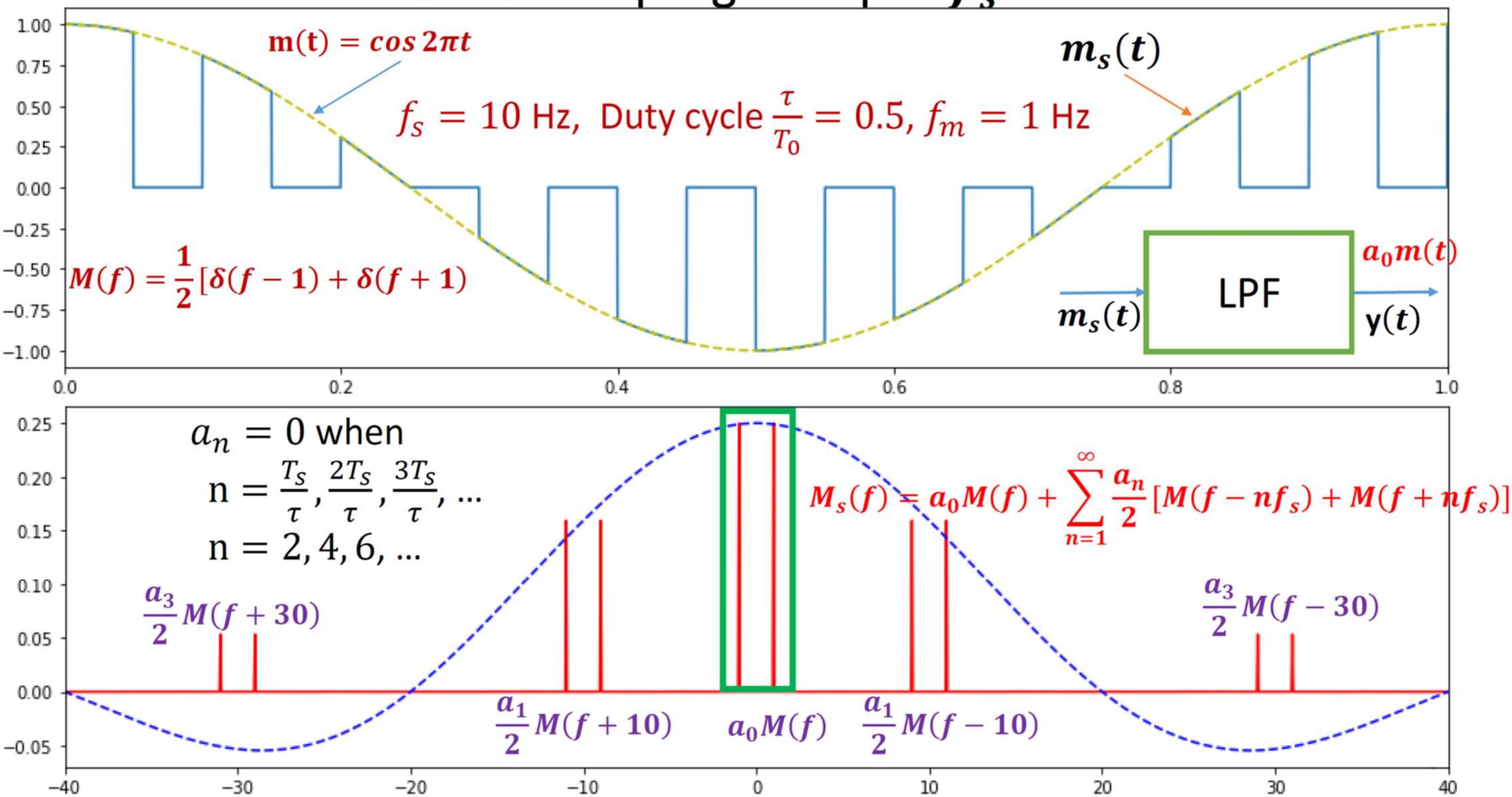
$Y(f) \neq a_0 M(f)$

$y(t) \neq a_0 m(t)$



Distortion due to aliasing is observed

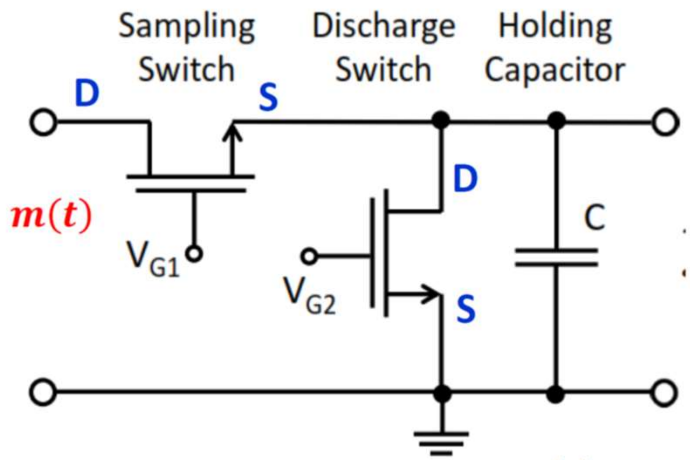
Natural Sampling Example: $f_s > 2W$



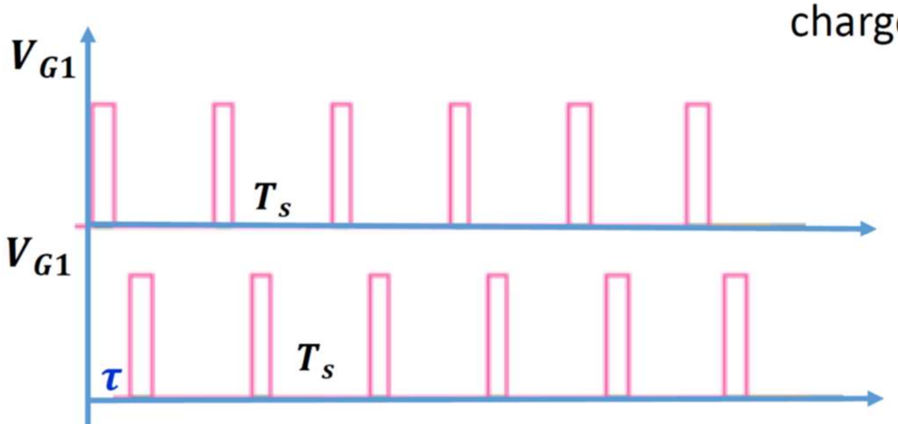
Flat-Topped Sampling
Lecture Outline

- This lecture continues the coverage of the techniques via which a continuous message signal can be sampled, as part of a PCM system
- **Sampling:** is the process by which a continuous time continuous amplitude signal is converted into a discrete time continuous amplitude signal.
- **There are three types of sampling:** Ideal sampling, natural sampling, and flat-topped sampling.
- Ideal sampling, natural sampling, the sampling theorem, and the phenomenon of aliasing were presented in the previous two lectures.
- In this lecture, we address the following topics
  - Flat-topped sampling (sample and hold).
  - Time division multiplexing (TDM)

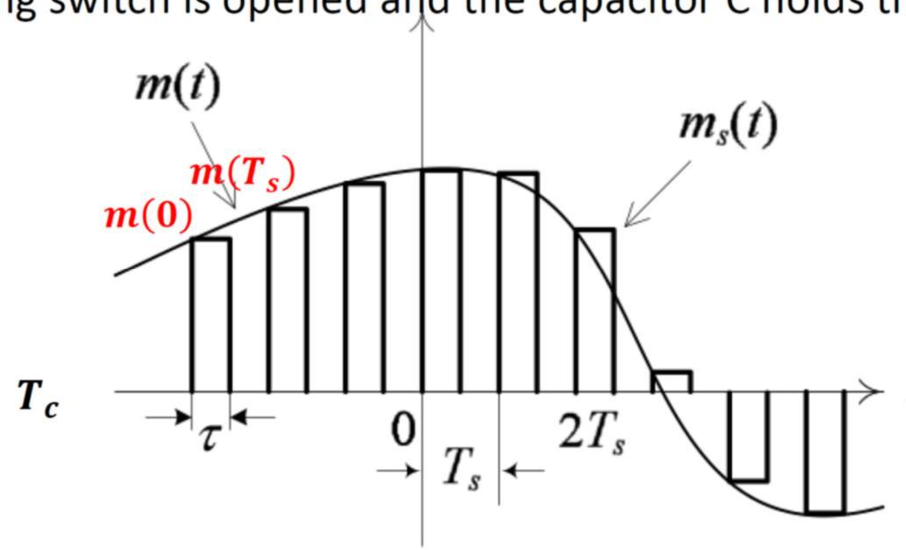
Flat-topped sampling (zero order hold sampling)



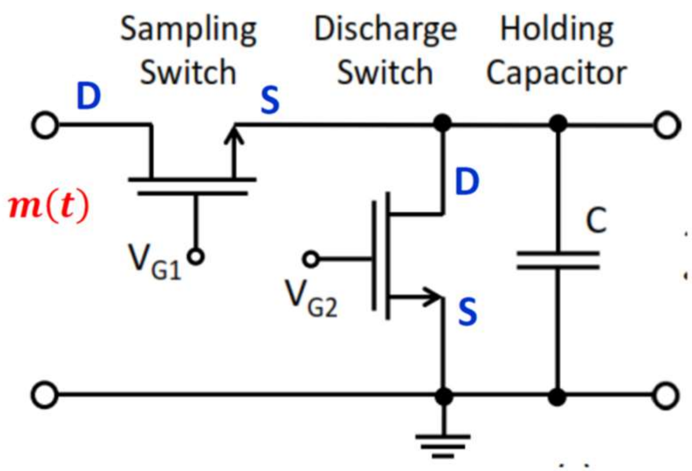
- The sample and hold circuit performs the task of sampling.
- The message m(t) is bandlimited to W Hz.
- The sample and hold circuit consists of two field effect transistor (FET) switches and a capacitor.
- The sampling switch is closed for a short duration by a short pulse applied to the gate G1 of the transistor. During this period, the capacitor C is quickly charged up to a voltage equal to the instantaneous sample value of the incoming signal m(t) .
- The sampling switch is opened and the capacitor C holds the charge.



Sampling frequency $f_s = 1/T_s$

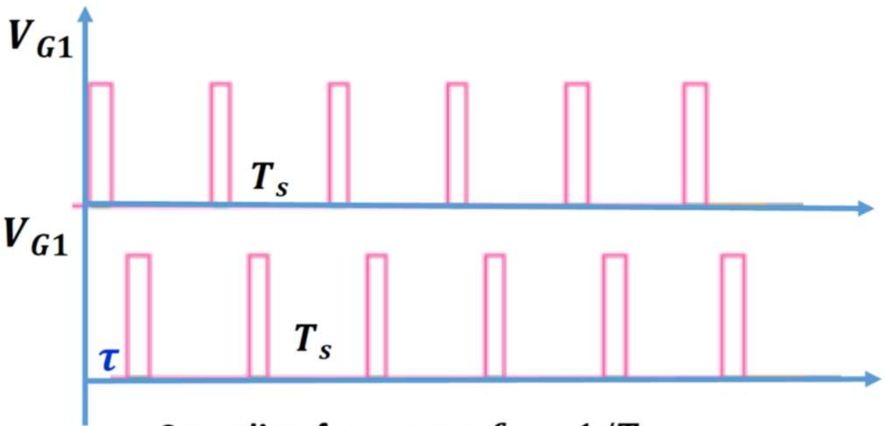


Flat-topped sampling (zero order hold sampling)

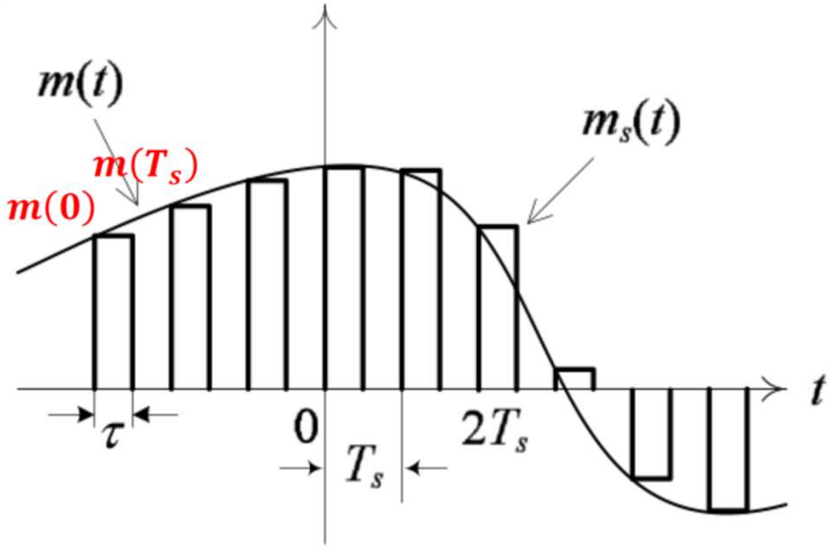


- The discharge switch is then closed by a pulse applied to its gate G2 at t = τ.
- Due to this, the capacitor C is discharged to zero volts.
- The discharge switch is then opened and thus the capacitor has no voltage.
- Hence the output of the sample and hold circuit consists of a sequence of flat topped samples

http://technical123b.blogspot.com/2016/11/pulse-amplitude-modulation-pam.html



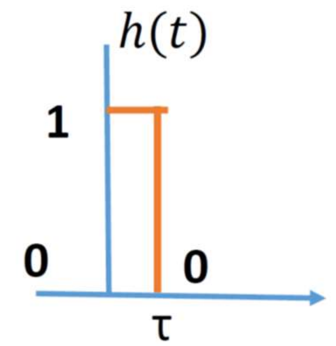
Sampling frequency $f_s = 1/T_s$

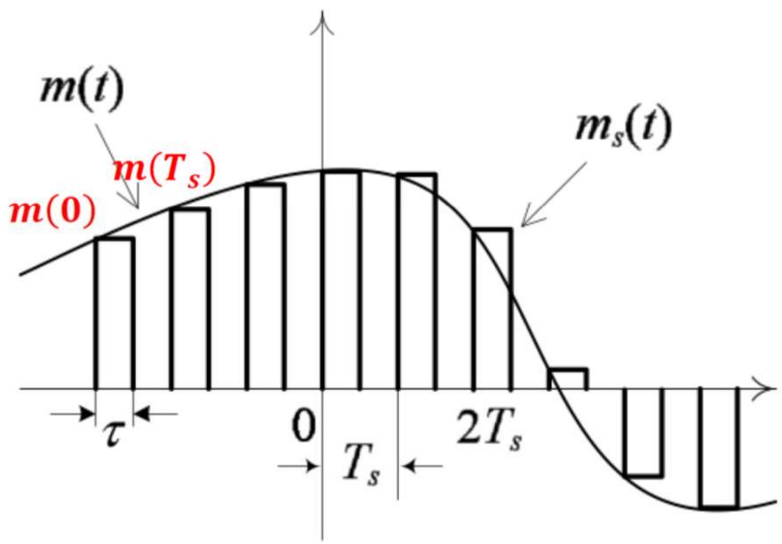


Flat-topped sampling: modeling

- Let h(t) be a basic unit amplitude pulse defined as:
- $h(t) = \begin{cases} 1 & 0 < t < \tau, \\ 0 & otherwise \end{cases}$
- In flat-topped sampling, the sampler generates a sequence of equally spaced rectangular pulses whose amplitudes are proportional to the message signal $m(t)$ at the sampling times $m(kT_s)$.
- The sampled signal is represented as

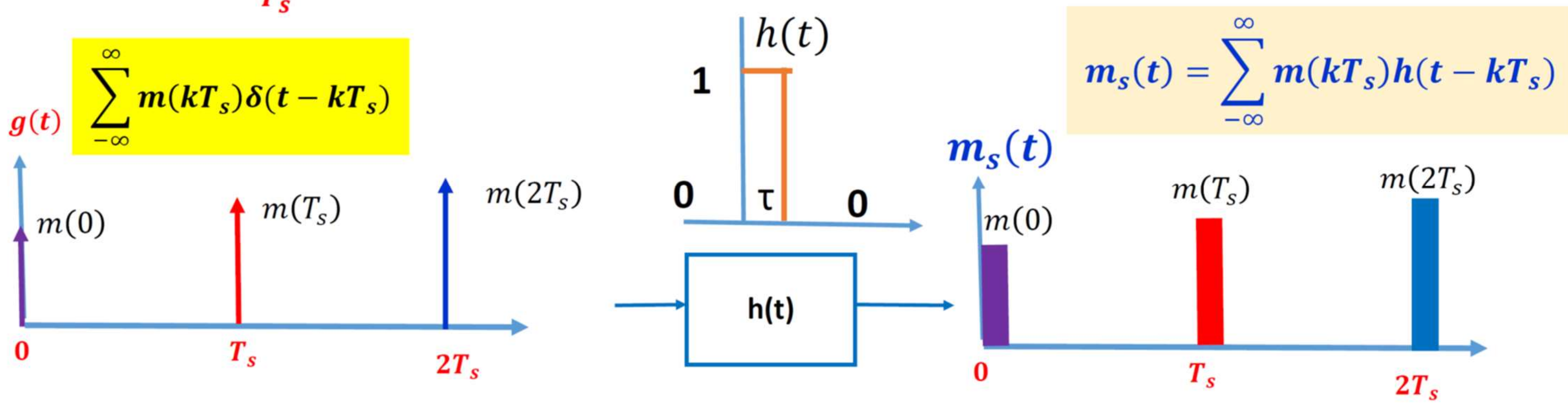
$$m_s(t) = \sum_{-\infty}^{\infty} m(kT_s)h(t - kT_s)$$





Flat-topped sampling: modeling

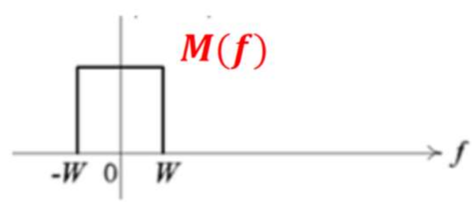
- The sampled signal is represented as $m_s(t) = \sum_{-\infty}^{\infty} m(kT_s)h(t - kT_s)$
- Using the identity, $\delta(t - kT_s) * h(t) = h(t - kT_s)$
- Multiplying both sides by $m(kT_s)$ we get, $m(kT_s)\delta(t - kT_s) * h(t) = m(kT_s)h(t - kT_s)$
- Therefore, $m_s(t)$ can be expressed as: $m_s(t) = h(t) * \sum_{-\infty}^{\infty} m(kT_s)\delta(t - kT_s)$
- Taking the Fourier transform, and recognizing that the second term corresponds to an ideally sampled sequence, we get
- $M_s(f) = H(f)\frac{1}{T_s}\sum_{-\infty}^{\infty} M(f - kf_s)$; where $H(f) = \tau sinc(f\tau)$.



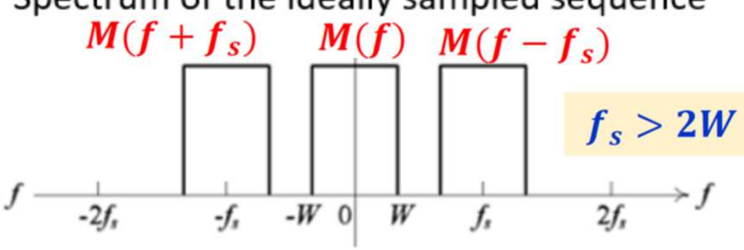
Flat-topped sampling: spectrum and message recovery

- $M_s(f) = H(f)\frac{1}{T_s}\sum_{-\infty}^{\infty} M(f - kf_s)$; $H(f) = \tau sinc(f\tau)$
- Here, we observe that the spectrum of the flat-topped sampled signal corresponds to the spectrum of the ideally sampled signal multiplied by the Fourier transform of the rectangular pulse $\tau sinc(f\tau)$.
- When $m_s(t)$ is passed through a low pass filter with bandwidth W, the output is y(t)

Spectrum of the message

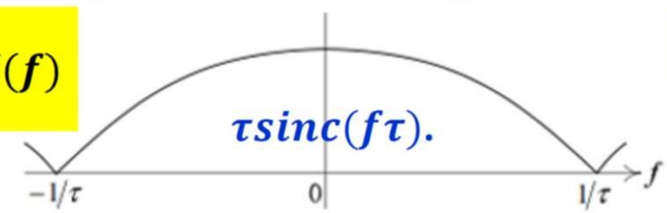


Spectrum of the ideally sampled sequence

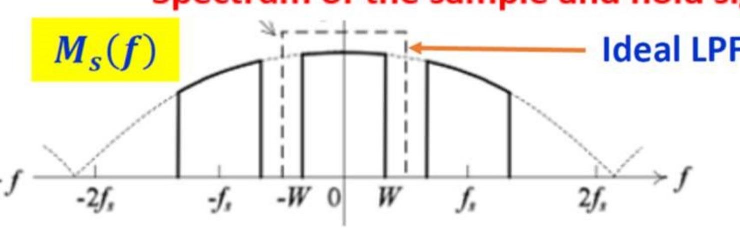


$$Y(f) = \frac{1}{T_s}\tau sinc(f\tau)M(f)$$

Spectrum of the rectangular pulse

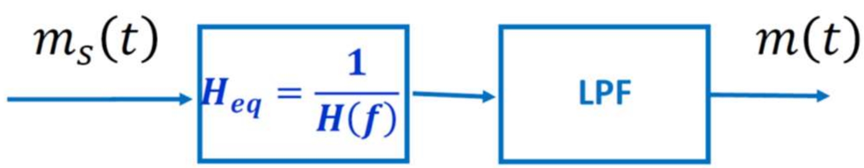


Spectrum of the sample and hold signal



Flat-topped sampling: equalization

- The LPF filter output is $Y(f) = \frac{1}{T_s}\tau sinc(f\tau)M(f)$.
- Note that $Y(f) \neq kM(f) \Rightarrow Distortion$.
- The distortion is due to the finite width τ of the sampling pulse.
- When the message B.W $W \ll 1/\tau$, the distortion is negligible. As τ increases, the distortion becomes more pronounced.
- Even though the signal is sampled at a rate $f_s > 2W$, a distortion is observed.
- A distortion-free signal can be obtained by using an equalizing filter whose transfer function is the reciprocal of the Fourier transform of the unit pulse

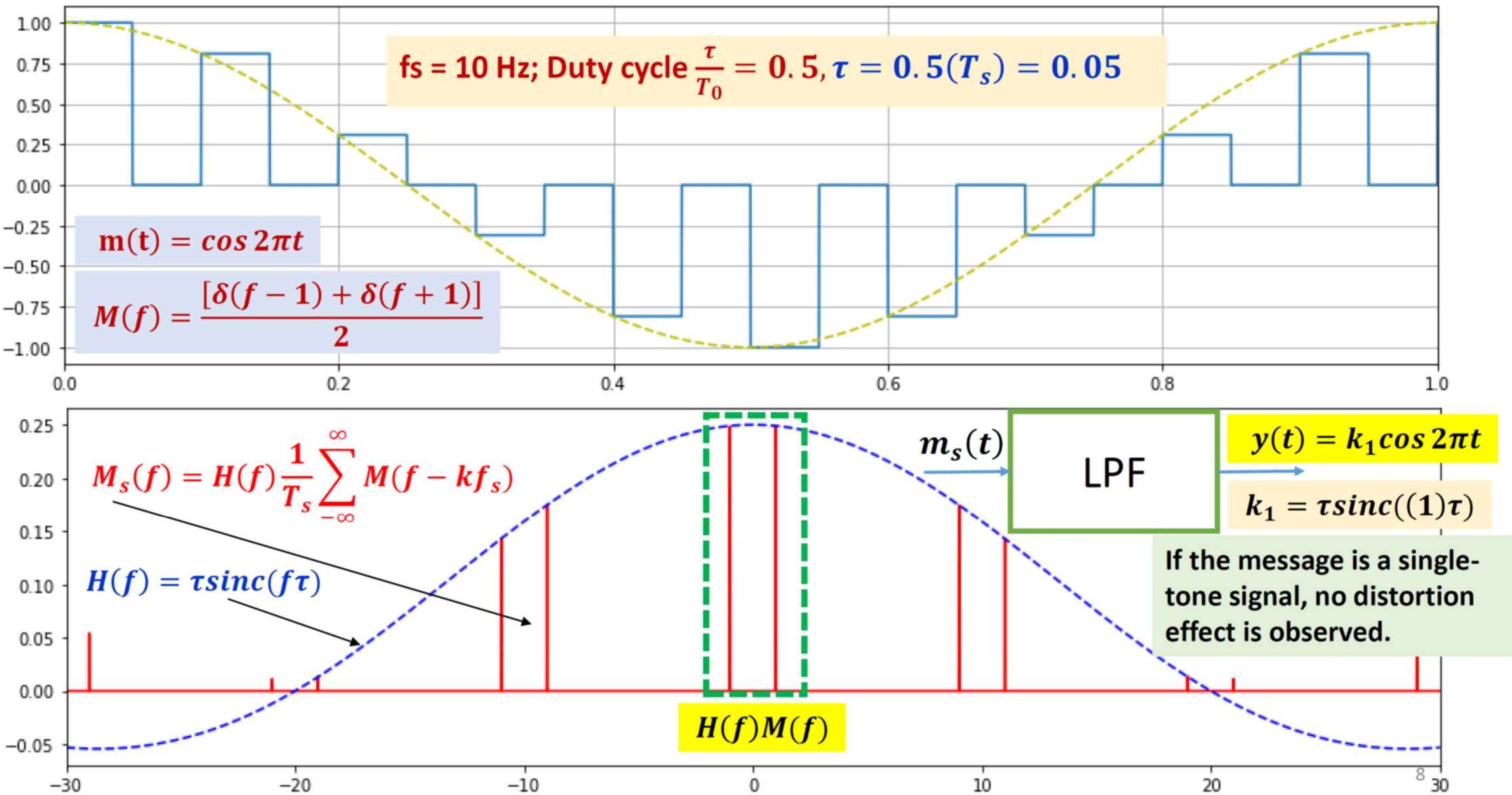


Here,

- $f_s > 2W$
- $H(f) = A\tau sinc(f\tau)$

Equalization implemented at the receiver

Flat-topped sampling: single tone example

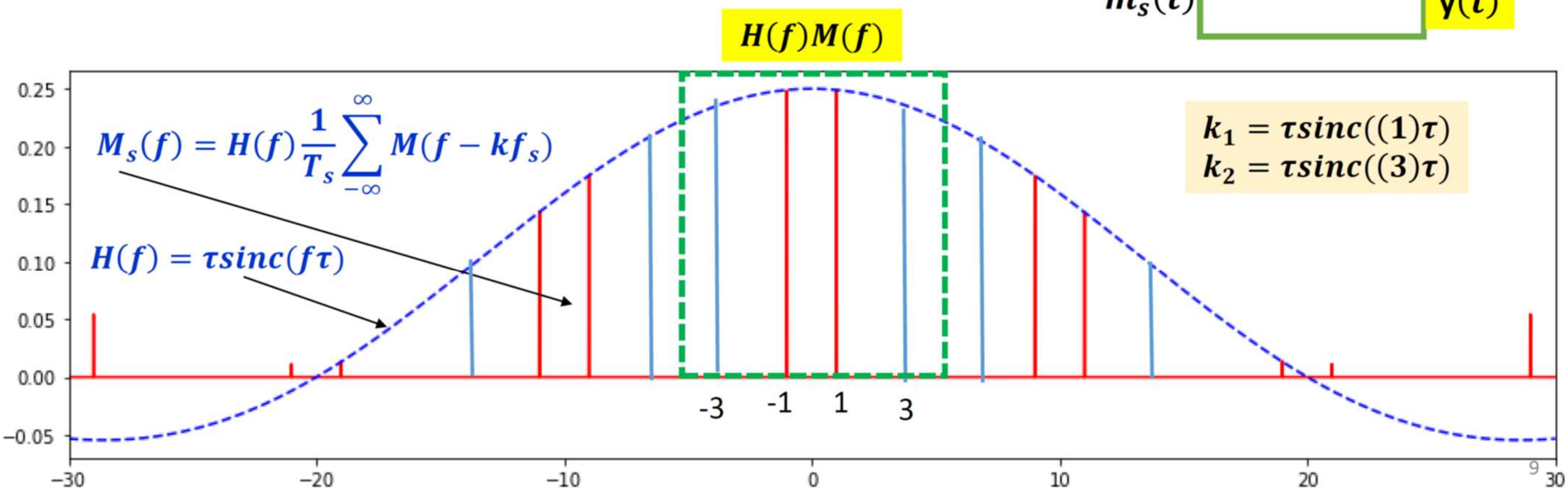


Flat-topped sampling: multi-tone example

Example: Repeat the previous example with fs = 10 Hz; Duty cycle $\frac{\tau}{T_0} = 0.5, \tau = 0.5(T_s) = 0.05$

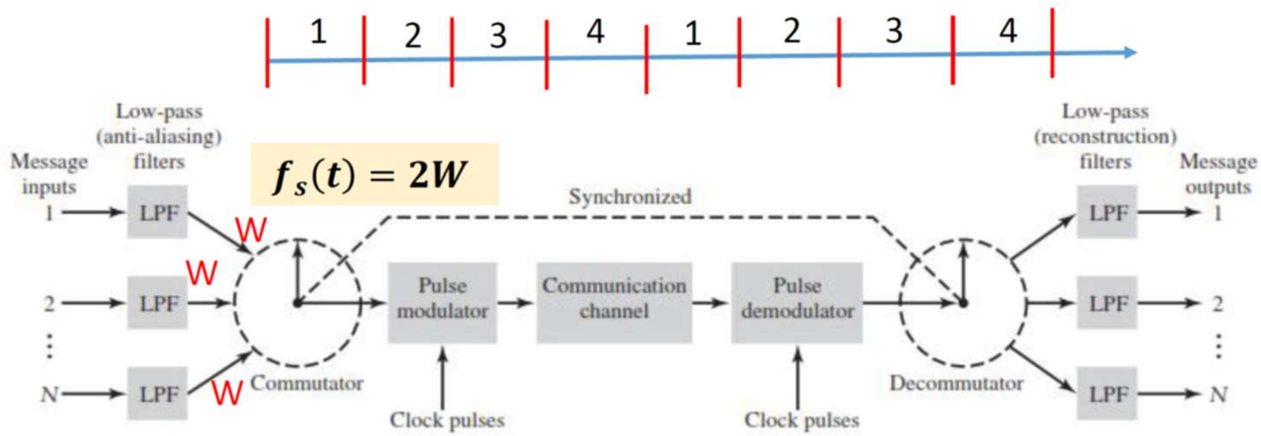
Now, let, $\mathrm{m(t)} = cos\,2\pi t + cos\,2\pi 3t$. $\boldsymbol{M(f) = \frac{[\delta(f-1)+\delta(f+1)]}{2} + \frac{[\delta(f-3)+\delta(f+3)]}{2}}$

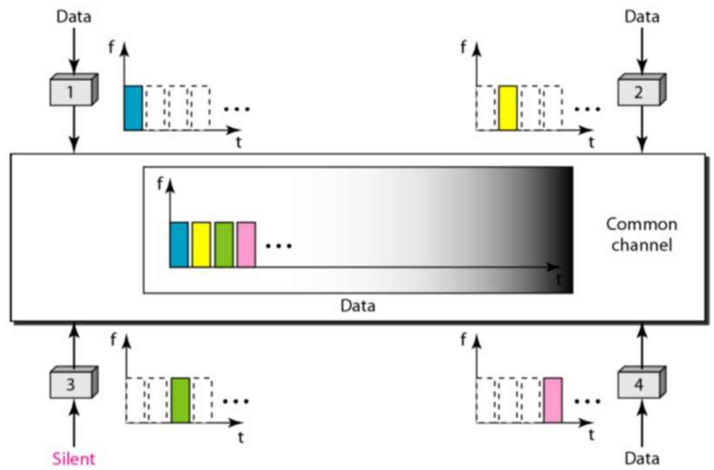
- The spectrum of the sampled signal is as shown in the figure below.
- The output of the LPF is $\boldsymbol{y(t) = k_1 cos\,2\pi t + k_2 cos\,2\pi 3t \neq km(t)}$
- Amplitude distortion is observed



Time Division Multiplexing (TDM)

- **Time Division Multiplexing (TDM)**: A technique which allows multiple users to use the same channel by assigning each user a portion of the transmission time without interfering with other users.
- Let N be the number of sources. The time axis is divided into N slots and each slot is allocated to a source.
- Each source transmits only during its slot, avoiding the possibility of a collision.
- When a user transmits during its slot, it utilizes the entire B.W. of the channel and this B.W. will be made available to the next user during the succeeding time slot.
- The collection of the N slots is called a **cycle**.
- TDMA requires some form of synchronization.
- The number of signal samples transmitted per second should be larger than the Nyquist rate.





Time Division Multiplexing: Example

$m_1(t) = cos\,2\pi t$ $\quad m_2(t) = \sin 2\pi t$ $\quad f_s(t) = 30$ $\quad f_{m1} = f_{m1} = 1$