Root Locus

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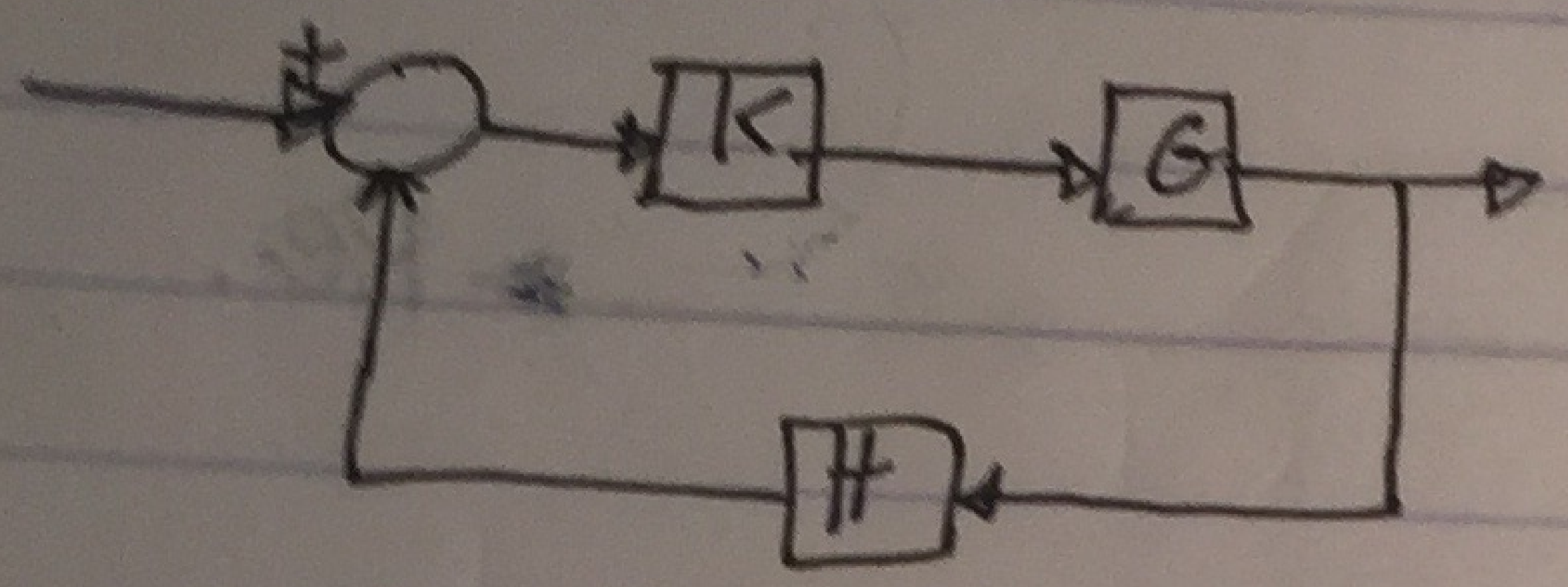
ROOT LOCUS TECHIQUES

Root locus : The set of all possible closed Loop poles that can be obtained by the variation of aproportial gain parameter with the open Loop function of the system .

$1+KGH=0 \quad , P=P(K)$.



Root locus :- هي مجموعة ال Poles الزي أحصل عليها من ال closed loop

$1+KGH=0$

$GH=\frac{-1}{K} \quad , \quad \bar{s}\in L \Longleftrightarrow GH(\bar{s})=\frac{-1}{K}$

Pole zero form.

$GH=\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}(s+P_i)}$ } open loop function. , $gh=\frac{\pi(1+\tau_{z_i}s)}{\pi(1+\tau_{P_i}s)}$. } → time constant form.

$GH(\bar{s})=\frac{-1}{K}$

شرط على الزاوية . $\angle GH=\begin{cases}(2\nu+1)\pi, & K>0\\ 2\nu\pi, & K<0\end{cases}$

شرط على القيمة $|GH|=\frac{1}{|K|}$ } عندما نحدد النقطة في ال locus فإننا بإمكاننا إيجاد قيمة K لحتى نصل إلى النقطة .

} Test on open loop function.

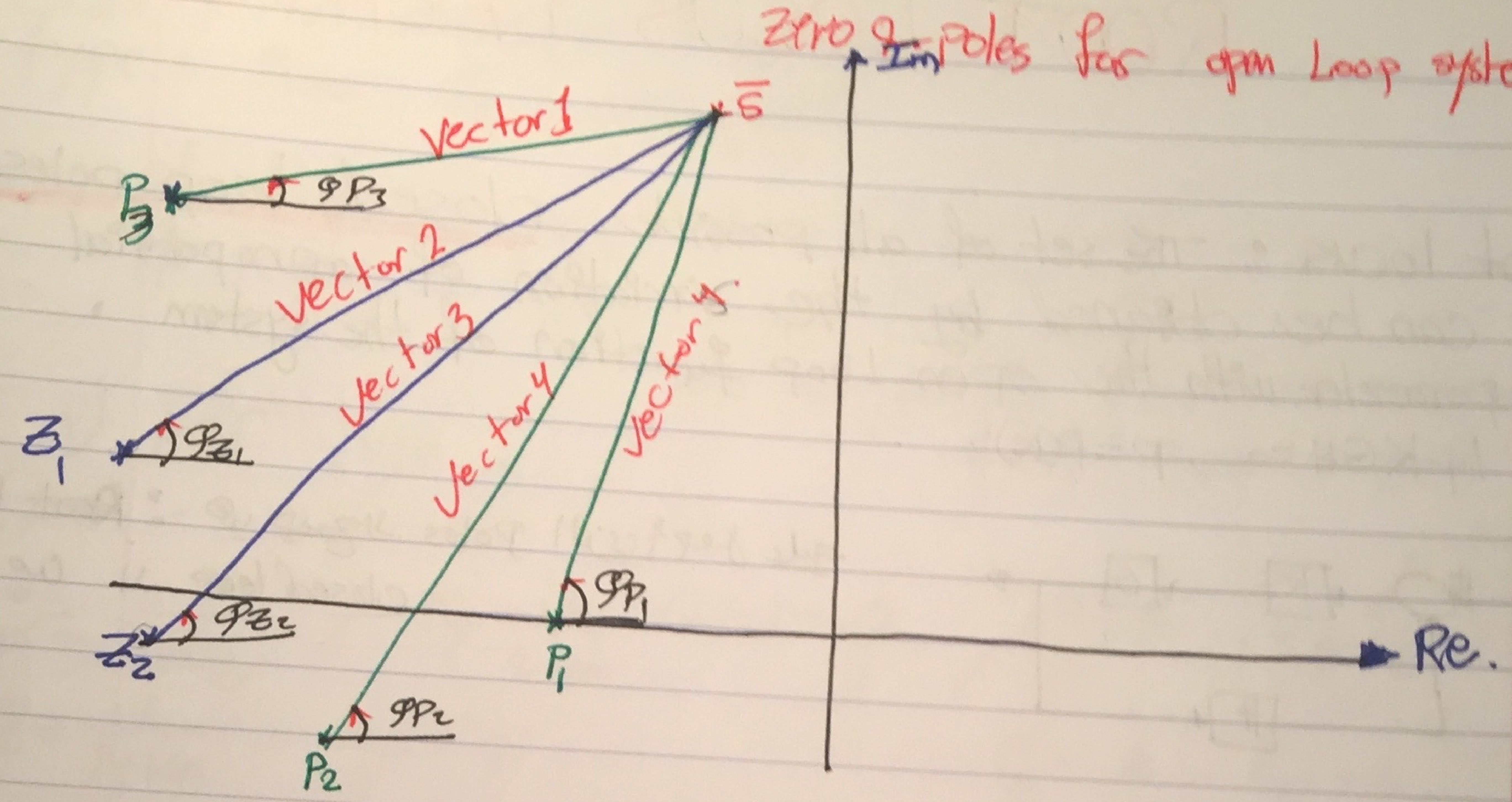
when $\bar{s}$: one of poles in root locuse.

every point satisfy the equation

$GH(\bar{s})=\frac{-1}{K}$

Mohamad Bomat

Zero & poles for open Loop system.



$$G(\bar{s})H(\bar{s}) = \frac{\prod_{i=1}^{m}(s+Z_i)}{\prod_{i=1}^{n}(s+P_i)}$$

لما نعاد pole في البداية م

جا زاوية بالنسبة لل s م

$\tan^{-1}(\varphi_z) - \tan^{-1}(\varphi_p)$

when $K > 0$

$$\varphi_{Z_1} + \varphi_{Z_2} - \varphi_{P_1} - \varphi_{P_2} - \varphi_{P_3} \overset{??}{=} (2\nu+1)\pi \begin{cases} \xrightarrow{\text{if yes}} \bar{s} \in L \\ \xrightarrow{\text{if No}} \bar{s} \notin L \end{cases}$$

when $K < 0$

$$\varphi_{Z_1} + \varphi_{Z_2} - \varphi_{P_1} - \varphi_{P_2} - \varphi_{P_3} = 2\nu\pi \xrightarrow{\text{if yes}} \bar{s} \in L$$

$$\frac{\prod \rho_{zi}}{\prod \rho_{pi}} = \frac{1}{|K|} \Rightarrow |K| = \frac{\prod \rho_{pi}}{\prod \rho_{zi}}$$

product for poles vector

product for zero vector

نحسب K لـ

الي ~~[illegible]~~

الي عالخط line لي

root locus.

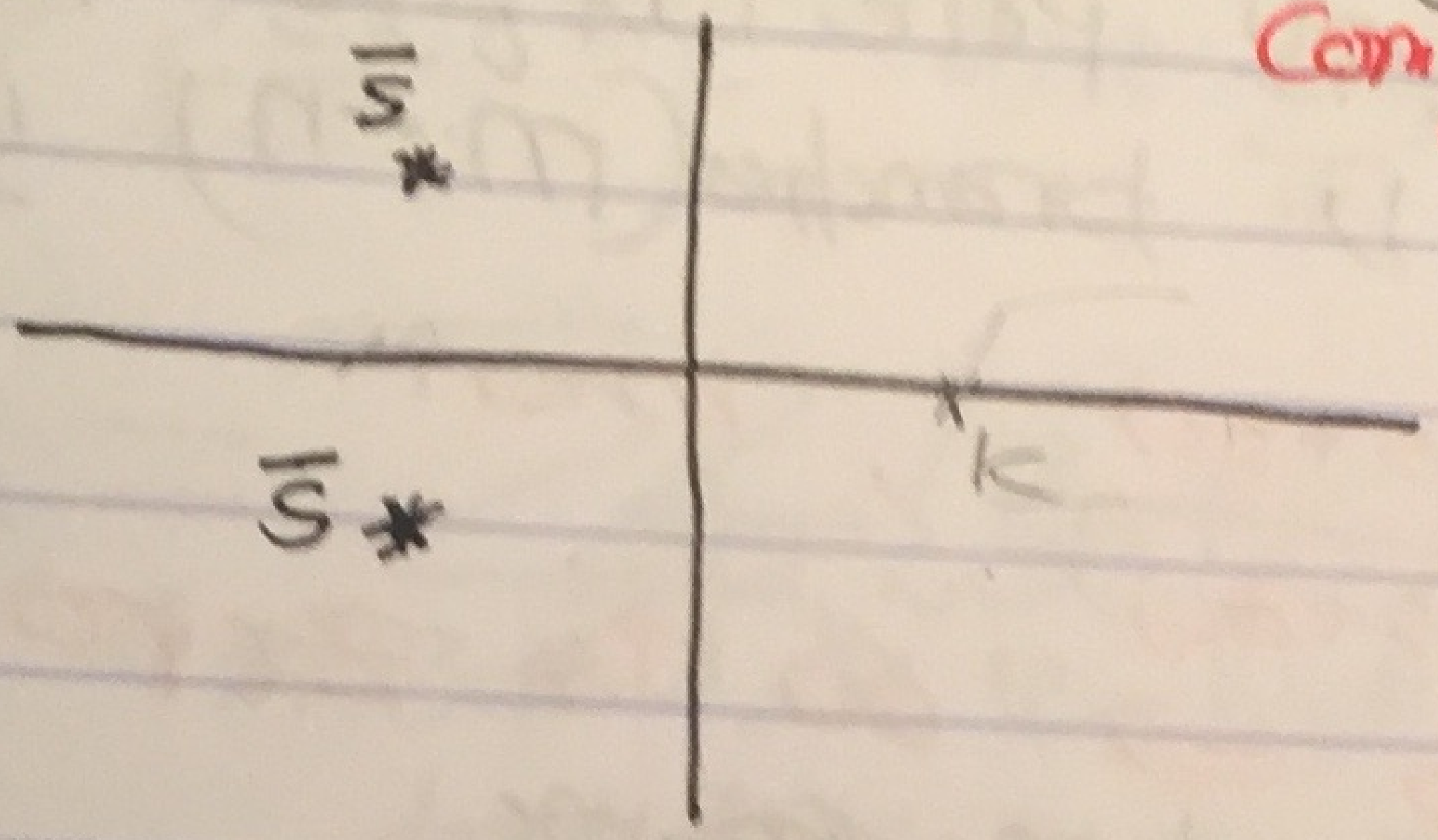
Properties of root Locus.

- The Root Locus has a number of branches equals to the maximume (m,n).

أكبر عدد من الـ zero أو الـ pole هو نفسه عدد الـ branches.

* The root Locus is symmetric with respect to real axis

إذا في كل pole بد complex pole and conjects.



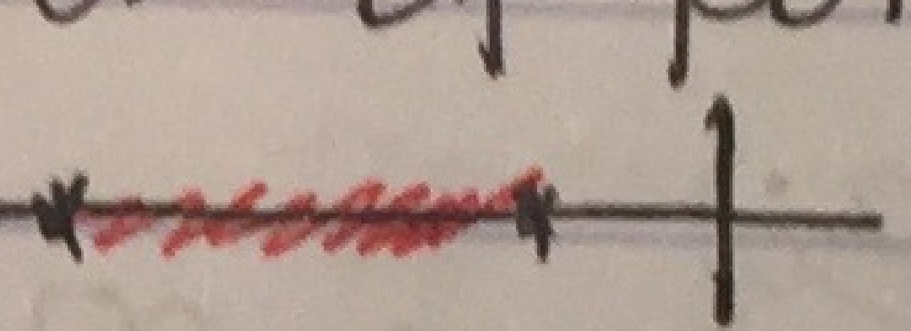
* The points of the real axise that belong to the Locous are Twise.

at real axis [illegible] Im [illegible]

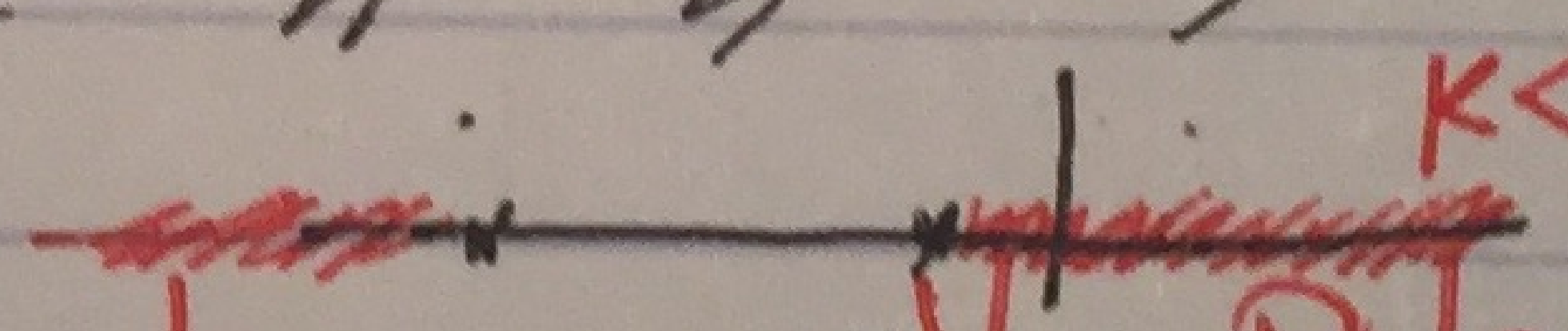
we have a complement on real axis.

الي بيتركوا على يمينهم عدد فردي من (الـ pole والـ zero).

1- That Leave $\overline{\text{an}}^{K>0}$ odd number of poles and zero to there right when $K > 0$.



2- That Leave an ~~odd~~ even " " " " " " " " " when $K < 0$.



* يجب أن نعد الـ pole والـ zero على اليمين

* each branch departs from apole and terminates at zero at finite or infinite.

$$T(s) = \frac{\prod (s + z_i)}{\prod (s + p_i)} \quad , \; T(s) = 0 \Rightarrow \text{zeros}$$

m branch $\longrightarrow$ m zeros at finite

$n-m$ $\xrightarrow{\text{zeros}}$ at infity asymptotically.

when $m>n$ [$m-n$ $\longrightarrow$ at infity poles.

هلا معنا ن على النقطة السابقة :

when $m < n$.

$$T(s) = \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)}$$

n = عدد ال branches

m = عدد ال zeros

كل branch يخرج من pole و يجب ان ينتهي في ال zero

و باقي ال (m−n) branches راح ينتهوا في ال ∞

لأنه $\lim_{s\to\infty} \frac{\prod_{i=1}^{m}(s+z_i)}{\prod^{n}(s+z_i)}$, $m<n$

$= 0 \Rightarrow$ ∞ is zero for $T(s)$

when $m > n$

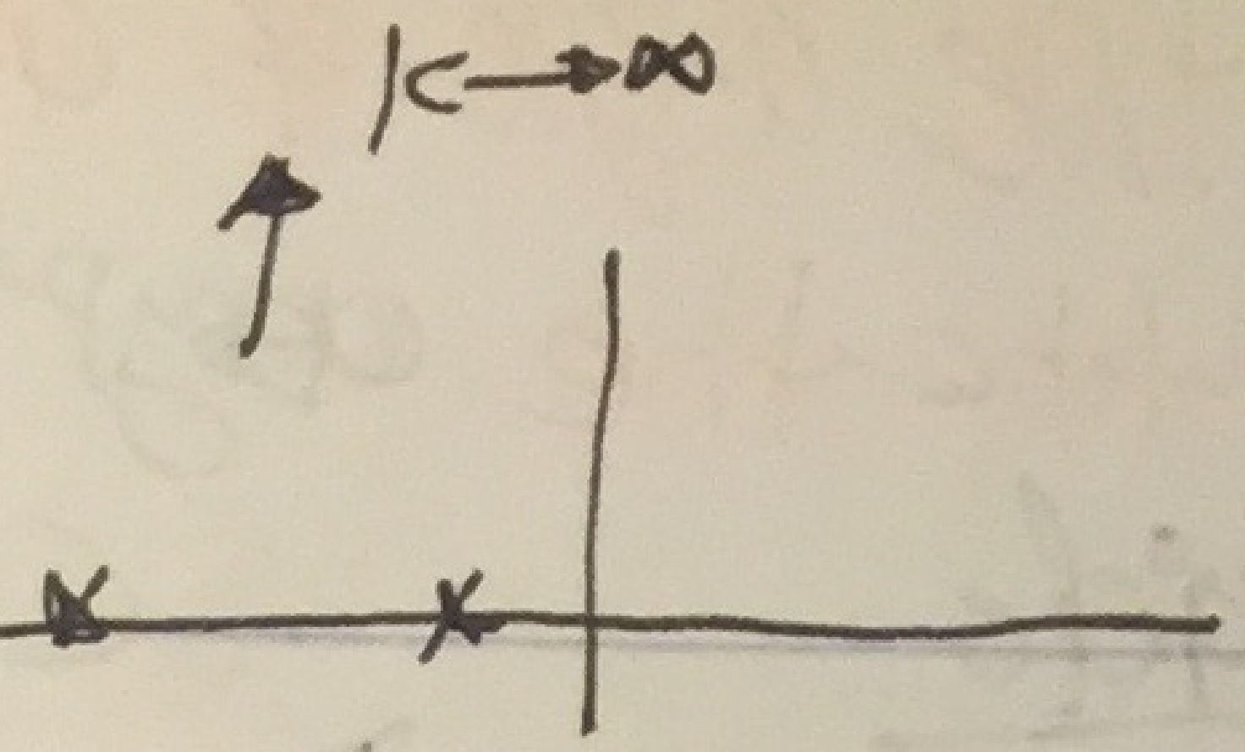
~~at ∞ (m−n)~~

at ∞ we have $(m-n)$ zero.

$$T(s) = \frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{i=1}^{n}(s+p_i)}$$

$$T(s) = \lim_{s\to\infty} \frac{\prod^{m}(s+z_i)}{\prod^{n}(s+p_i)} = \infty$$ { at ∞ we have $(m-n)$ poles.

- When we have ~~[illegible]~~ Two poles.



الزاوية التي ينطلق فيها .

- Calculation of departion from apole P_i & Arraival to a ~~pole~~ zero ~~Z_i~~

for departion { $\varphi_{P_i} = \sum_{K=1} \measuredangle (P_K + P_i) + \sum_{K=1} \measuredangle (Z_K + P_i) = (2\nu+1)\pi$ } for $K>0$

$$\measuredangle P_i = \sum \measuredangle (Z_K + P_i) - \sum \measuredangle (P_K + P_i) + (2\nu+1)\pi$$

الزاوية التي ينطلق فيها .

الزوايا بين الـ P_i و الـ zeros الموجودة المعتبرة نحسبها عن طريق $\tan^{-1}\left(\frac{Im}{re}\right)$

الزوايا بين P_i والـ poles نحسبها عن طريق $\tan^{-1}\left(\frac{Im}{rel}\right)$.

for arraival

$$\varphi_{Z_i} = \sum \measuredangle (P_K + Z_i) - \sum \measuredangle (Z_K + Z_i) + (2\nu+1)\pi$$

~~Zero الـ Zero الـ [illegible] الزاوايا بين~~

الزوايا بين الـ poles والـ zero المقصود

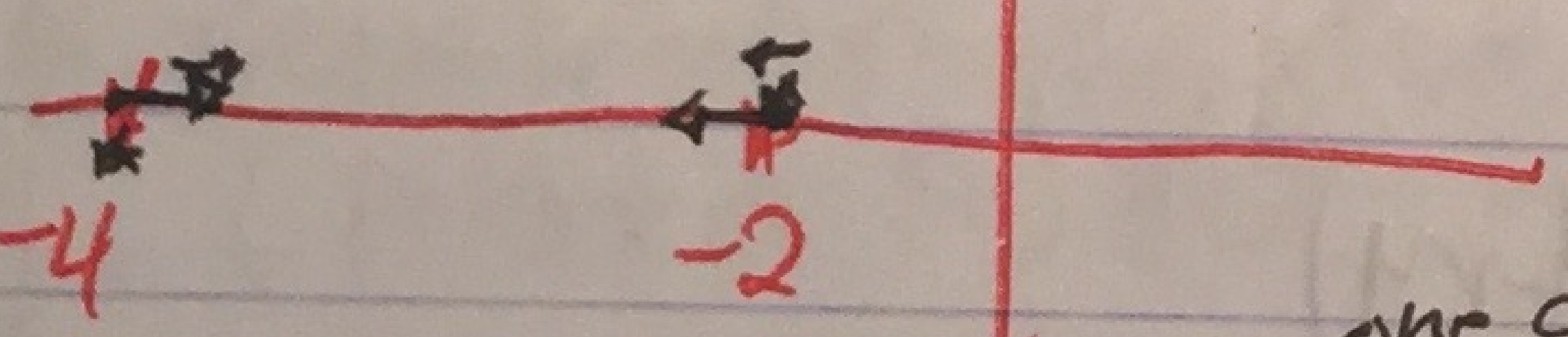
~~والـ pole الـ~~

~~[illegible] الزاويا بين الـ~~

~~الـ zero والـ zero المعتبر~~

الزوايا بين الـ zeros

والـ zero المقصود



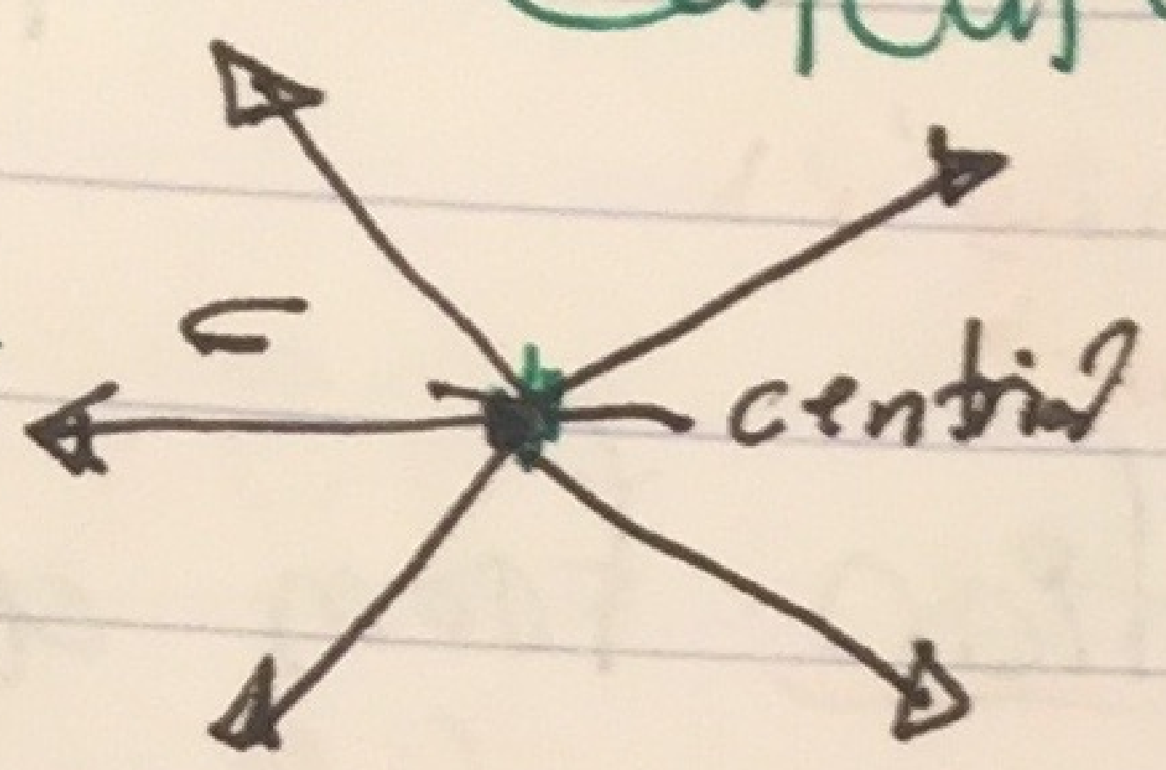
one cycle.

$\varphi_{-2} = 0 + 0 + (2\nu+1) \cdot \pi = \pi$

$\varphi_{-4} =$ ~~[illegible]~~

"Winter is Coming"

-when the angle of arrival at infinity, we want to calculate the assyptitic. (الزوايا التي بيعملها عند ال ∞)



Centraide

- Calculation the asymptotes.
(Direction)

Where the assymptotic cross each other.

نقطة الإلتقاء.

$n > m$

إلتقاء ال asyptote.

$$\boxed{\sigma} = \frac{\Sigma P_i - \Sigma Z_i}{n-m}$$

when $K>0$

$$\varphi_\nu = \frac{(2\nu+1)\pi}{n-m}, \quad \nu = 0, 1, . n-m-1$$

$K>0$.

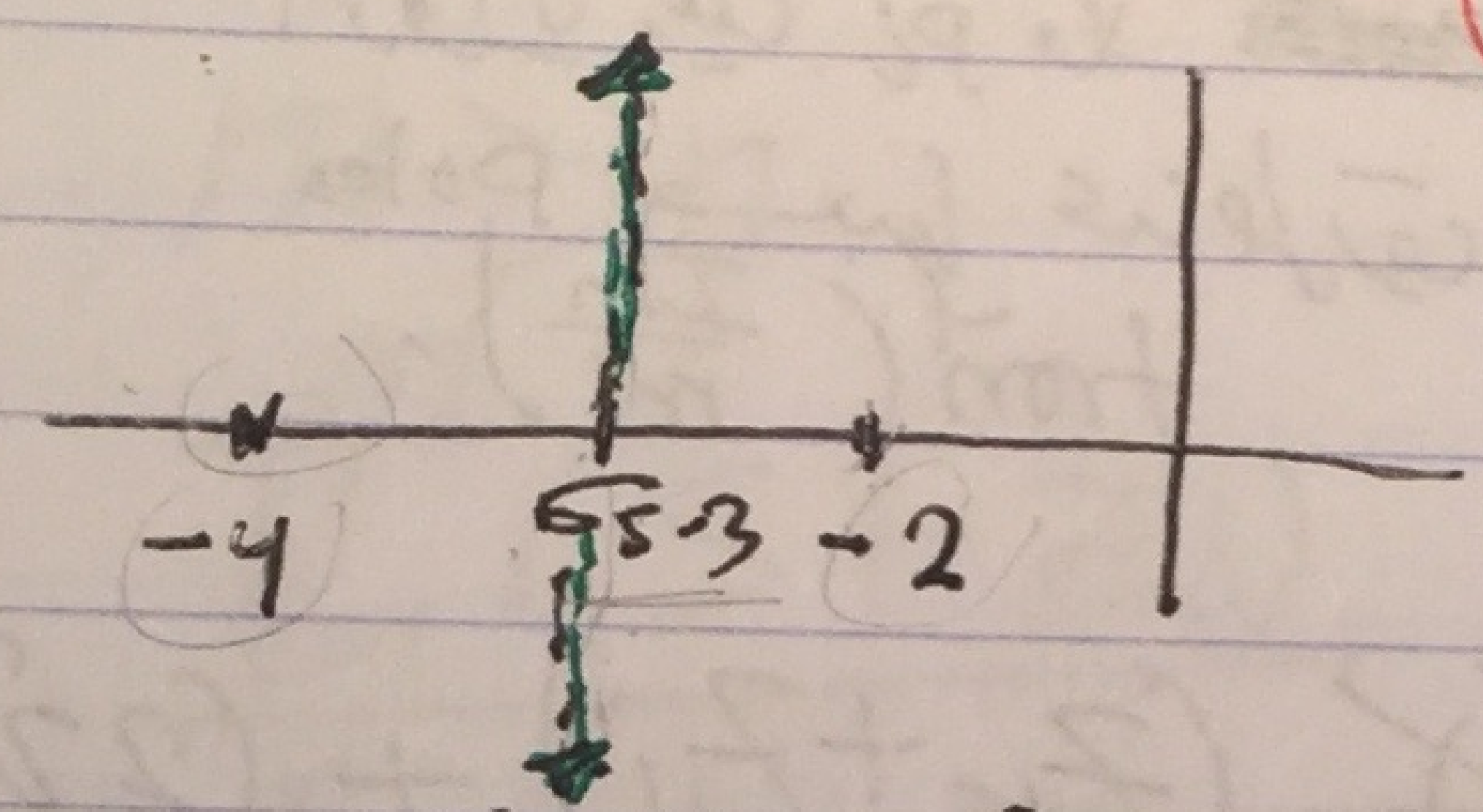
→ for ~~arrival~~ departion. to infinite (not to zero)

(زاوية ال assyptotic على ال infinity. φ_ν)

→ When $K<0$

$$\sigma = \frac{2\nu\pi}{n-m}.$$

Ex



$$\sigma = \frac{-2-4-(0)}{2} = -3.$$

$$\varphi_0 = \frac{(2\nu+1)\pi}{2} = \frac{\pi}{2}.$$

$$\varphi_1 = -\frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$ (By symmetry)

~~Branch~~

To calculate where we have a point branches.

(نقطة التفرع)

repeated poles.

$$F = 1 + GH = 0$$

$$\frac{dF}{ds} = 0 = \frac{dGH}{ds}$$

$$\frac{d^2F}{ds}$$

→ repeated Solution.

$$\frac{d^{n-1}}{}=0$$

or by.

$$\sum_{i=1}^{n}\frac{1}{(S_d+P_i)}-\sum_{i=1}^{m}\frac{1}{(S_d+Z_i)}=0$$

قيمة ال pole

$$\frac{1}{s+2}+\frac{1}{s+4}-0=0$$

The calculation by 1+GH.

$(s+4+s+2)=0 \rightarrow \boxed{s=-3}$ } the point of branches.

2-solutions إذا كان يوجد
يجب أن تأخذ التي تنتمي . وهنا ان
zasmpol[illegible] و يوجد

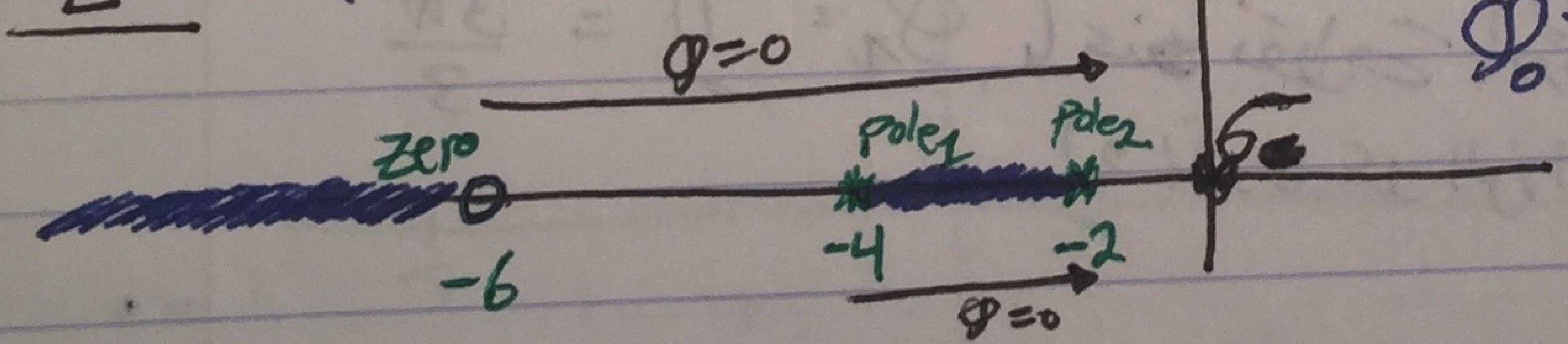
- ليس شرطاً أن تكون σ تنتمي للـ Locus.
- نقطة الـ repeated poles هي نفس (branches التقاء الـ
نقطة الـ Centroid (التقاء الـ asymptotic) الذي تذهب للامالانهاية
ولكن الذي يشترط أن تكون في الـ Locouse.
هو S (الـ repeated pole)

$\sigma_c \in I$ not necessary.

$S_d \in L$.

Ex

يجب أن تترك على يمينها عدد فردي
من الـ (poles+zeros)



كم زاوية للـ

ν (0, 1, 2 … (n−m)−1].

$\phi_0 = \frac{(2\nu+1)\pi}{1} = \pi$ (the angle of departion to [illegible]).

- each region have one repeated pol

1^{st} between (−4 → −

2^{nd} between (−∞ → −

$$\sigma_c=\frac{\sum P_i-\sum Z_i}{n-m}=\frac{-2-4+6}{1}=0$$

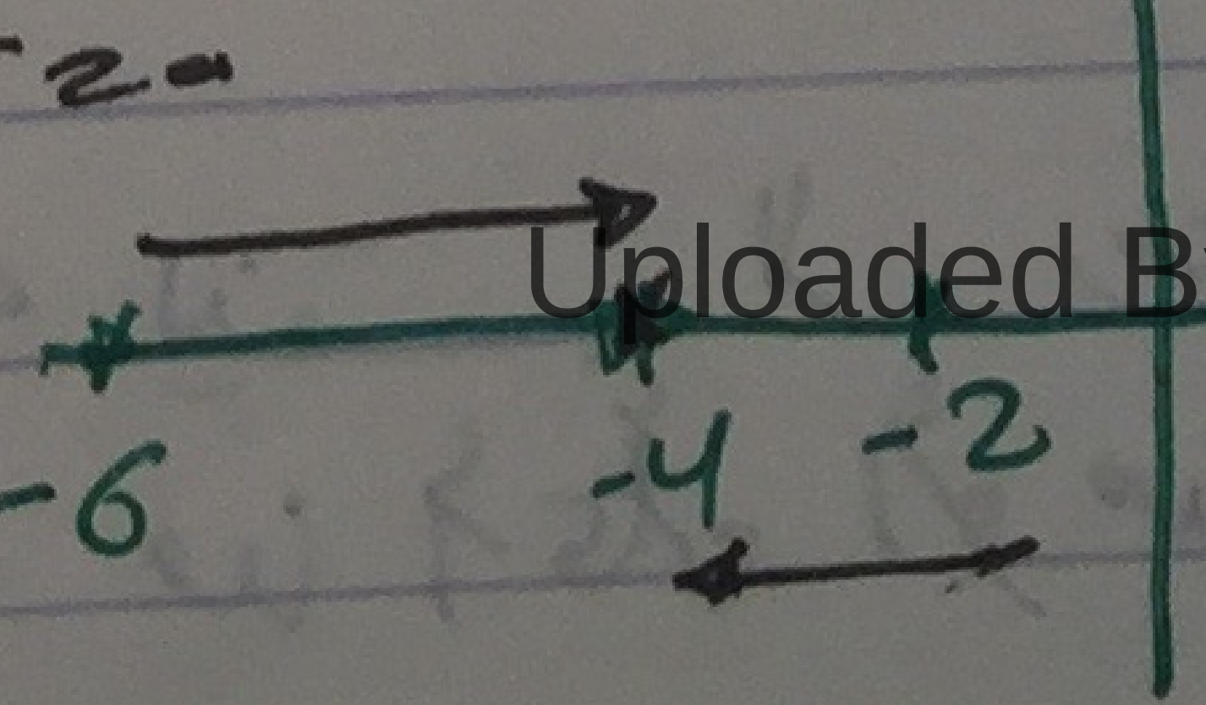
صفر الـ Locouse ، ليس مشكلة.

$\phi_{-2} = 0 - 0 \pm (2\nu+1)\pi = \pi$ (so depaotion at −2 = π).

−2 pole

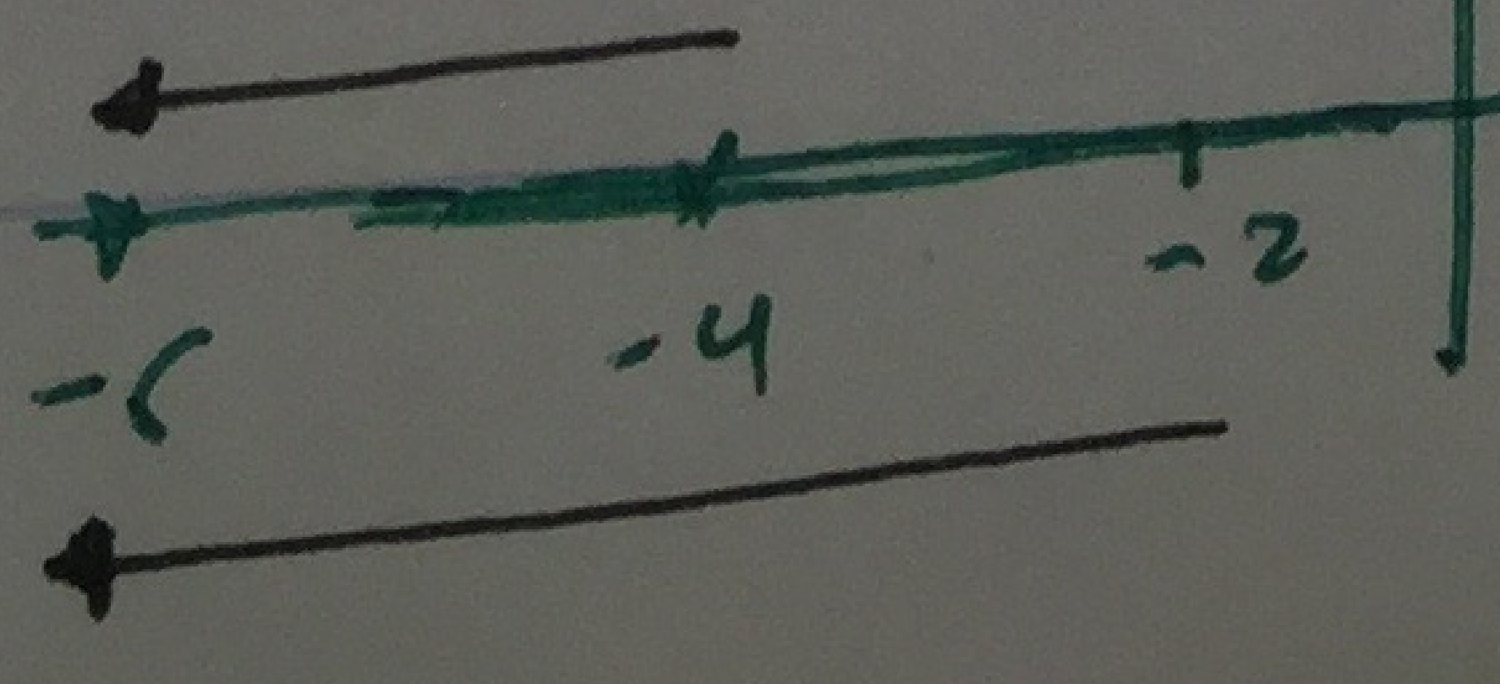
$\phi_{-4} = 0 - \pi + (2\nu+1)\pi = 0$

(the angle of departion at −4 pole



the angle of arrival for −6

$\phi_{-6} = +\pi+\pi+\pi = 3\pi$



كم طريقة نقدر نحل critical solution
هو نفسه مثل ال
repeated pole.

- TO find the (Double). repeated poles.

$$\frac{1}{s+2}+\frac{1}{s+4}-\frac{1}{s+6}=0$$

$$(s+4)(s+6)+(s+2)(s+6)-(s+4)(s+2)=0$$

$$s^2+12s+28=0$$

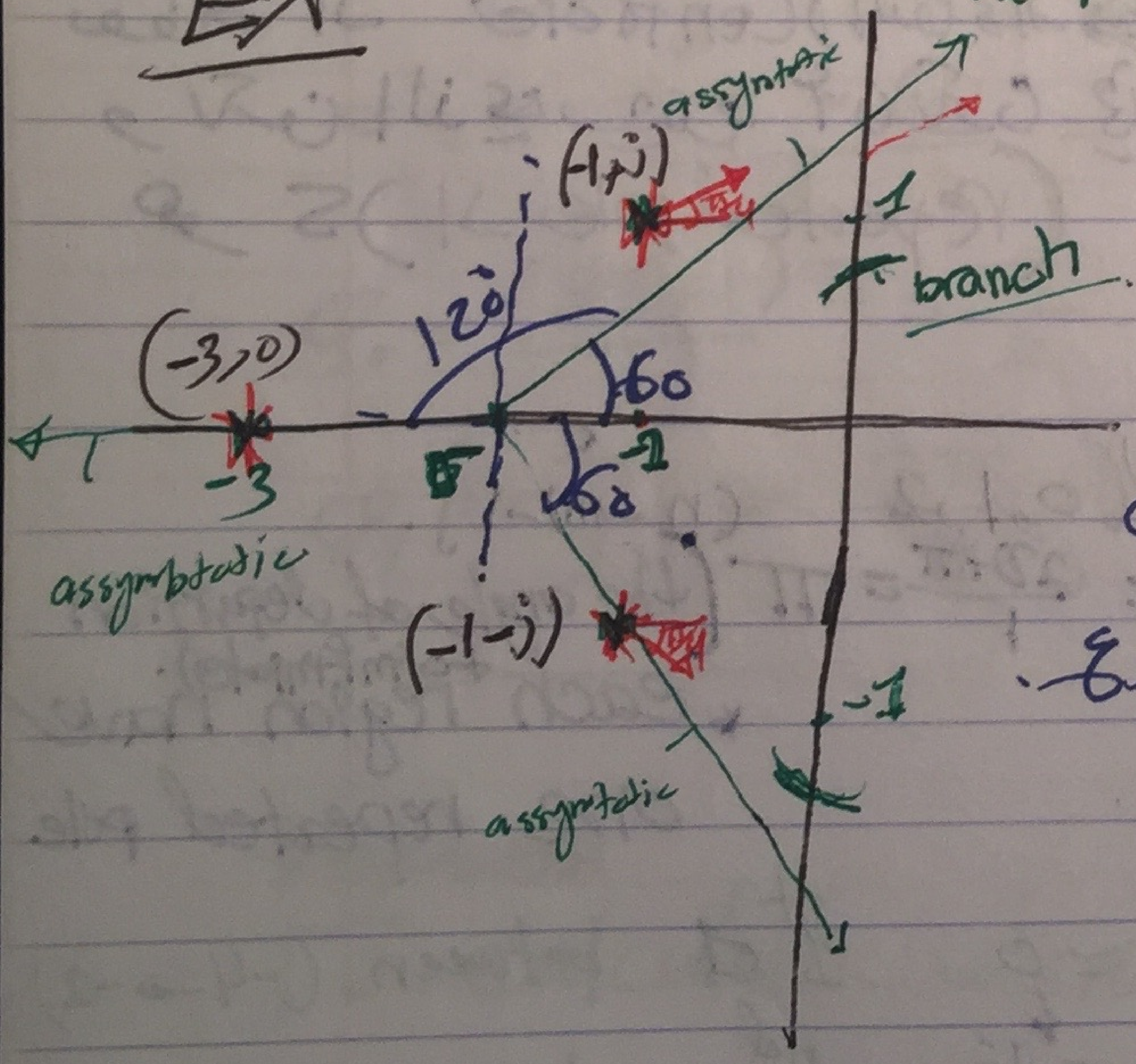
$S_1=-3.178$, $S_2=-8.825$ → repeated pole.

$2*\overset{2}{P}=4$ = ص , خاصية :- عدد ال sector = $2*\#$Poles

$\frac{2\pi}{\#\text{ of sector}}=\frac{2\pi}{\#\text{ of poles}}$ = وبالتالي الزاوية لكل sector

repeated pole.

EX



$$\sigma_c=\frac{-1-1-3}{3-0}=\frac{-5}{3}$$

الزوايا التي تخرج منها ال asyptotic عند نقطة التفرع.
(ويكون عدد الزوايا)

$$\left\{\begin{array}{l}\varphi_0=\frac{\pi}{3}\\ \varphi_1=\frac{-\pi}{3}\\ \varphi_2=\pi=\frac{3\pi}{3}\end{array}\right.$$

كل ما التقى 2-branches فإن نقطة الخروج من الالتقاء فإنه يقسم
ال space لـ $2n_0$ sector

- عندما يكون عندنا 2 ـــ فإنه يقسم ال space لـ 4-sector
- لحتى نحدد ال branch ـ ب ـ من أي شكل او من تحت ال asyptotic.

$$1+\frac{K}{(s^2+2s+2)(s+3)}=0 \qquad (1+KGH=0)$$

$(s^2+3s+2)(s+3)+K=0$

$s^3+5s^2+8s+6+K=0$

to Knao where crossing the imaginary axis.

3	1	8	0
2	5	6+K	0
1	34-K	0	
0	6+K	0	

when $K=34$

or $K=-6$ ✗

إذ عند K بصير أن [illegible]

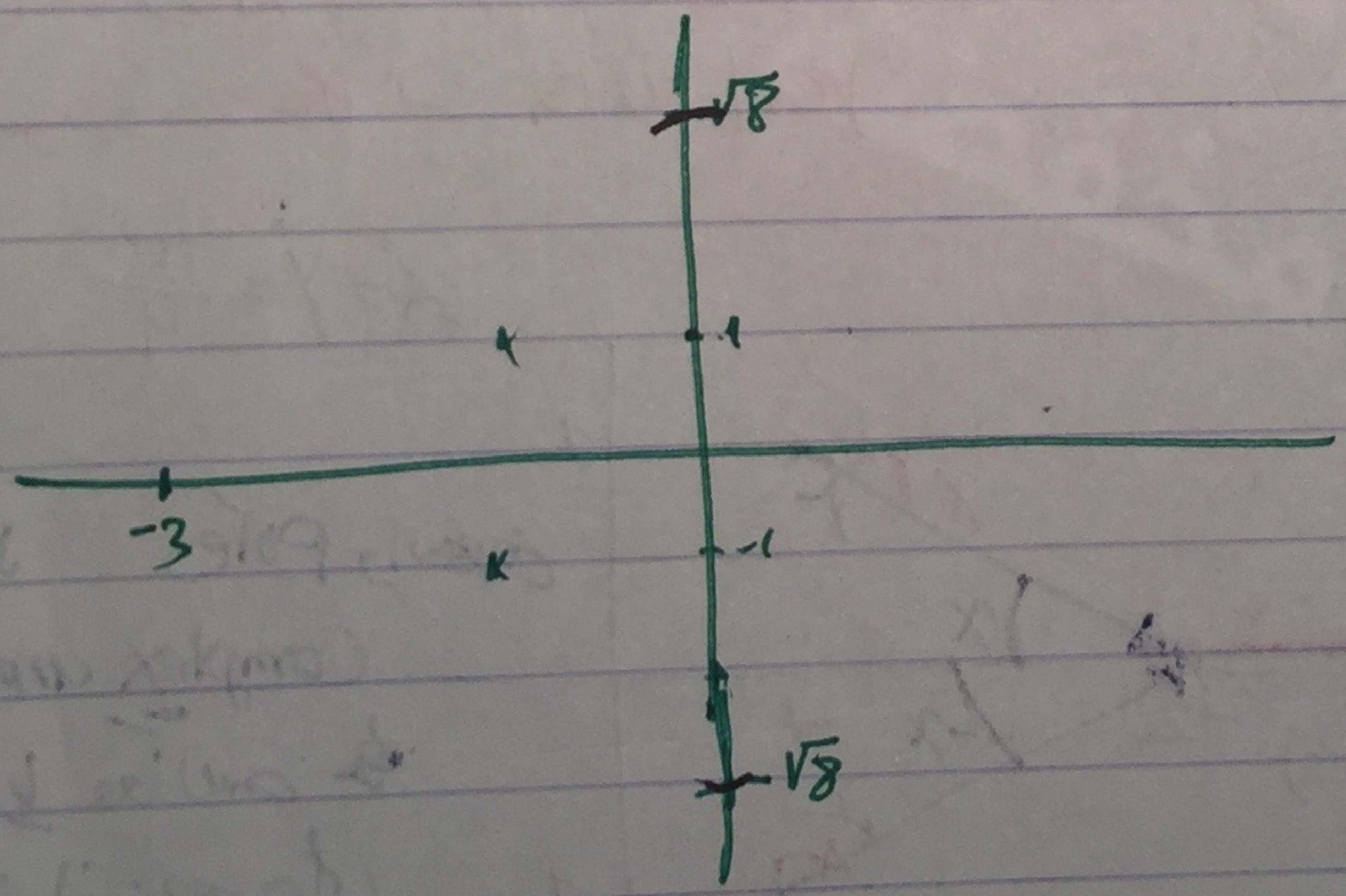
so $K=34$

Auxiliary polynomial

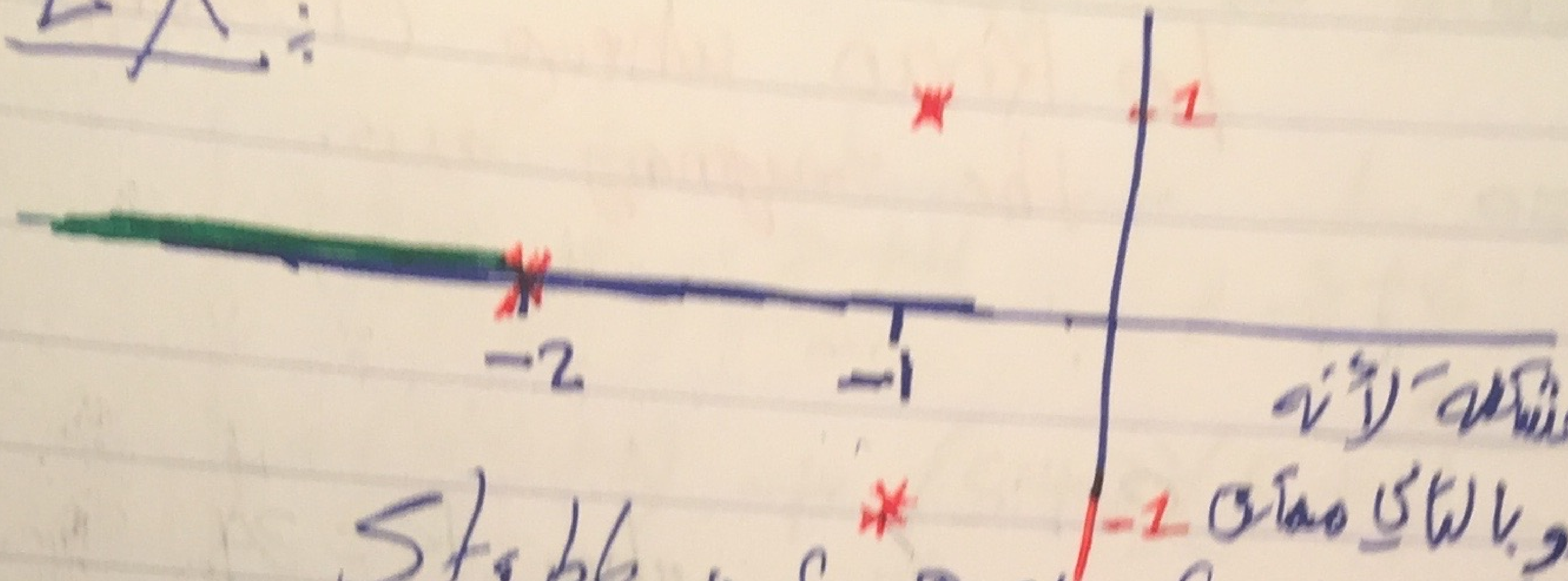
$= 5X^2+6+K = 5X^2+6+34 = 5X^2+40$

$= 5s^2+40$

$s=\mp j\sqrt{8}$



EX :



$n > m$

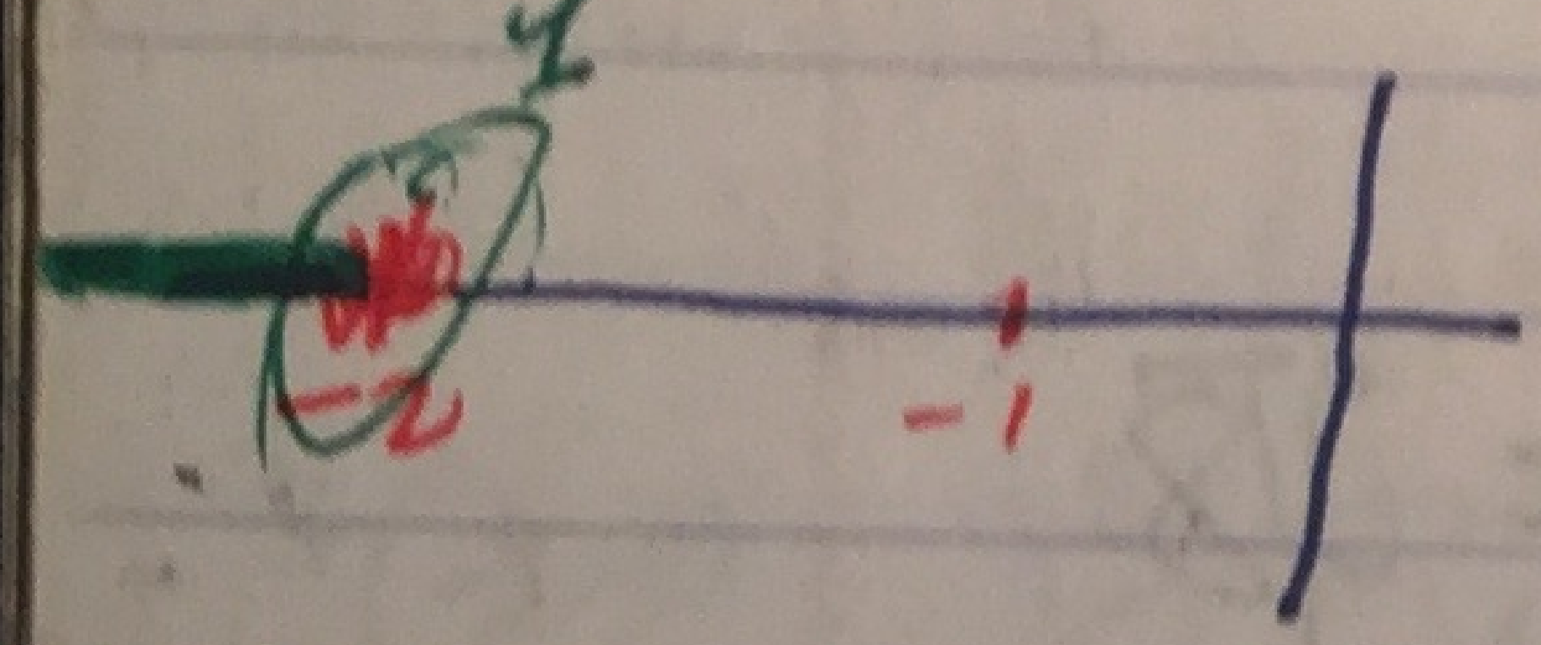
- لو كان ال pole على اليمين ليس مشكلة لأنه
هذا ال poles ال openloop و بالتالي ممكن
أن يكون ال openloop (unstable) وال closed loop هو يكون Stable

☒ Number of ~~branches~~ = number of poles = 3.
(max between poles and zeros)

☒ The root Locus is symmetric with respect to the real axis.

☒ points of the real axis that belong to the Locus

$$S \in \left]-\infty, -2\right[$$

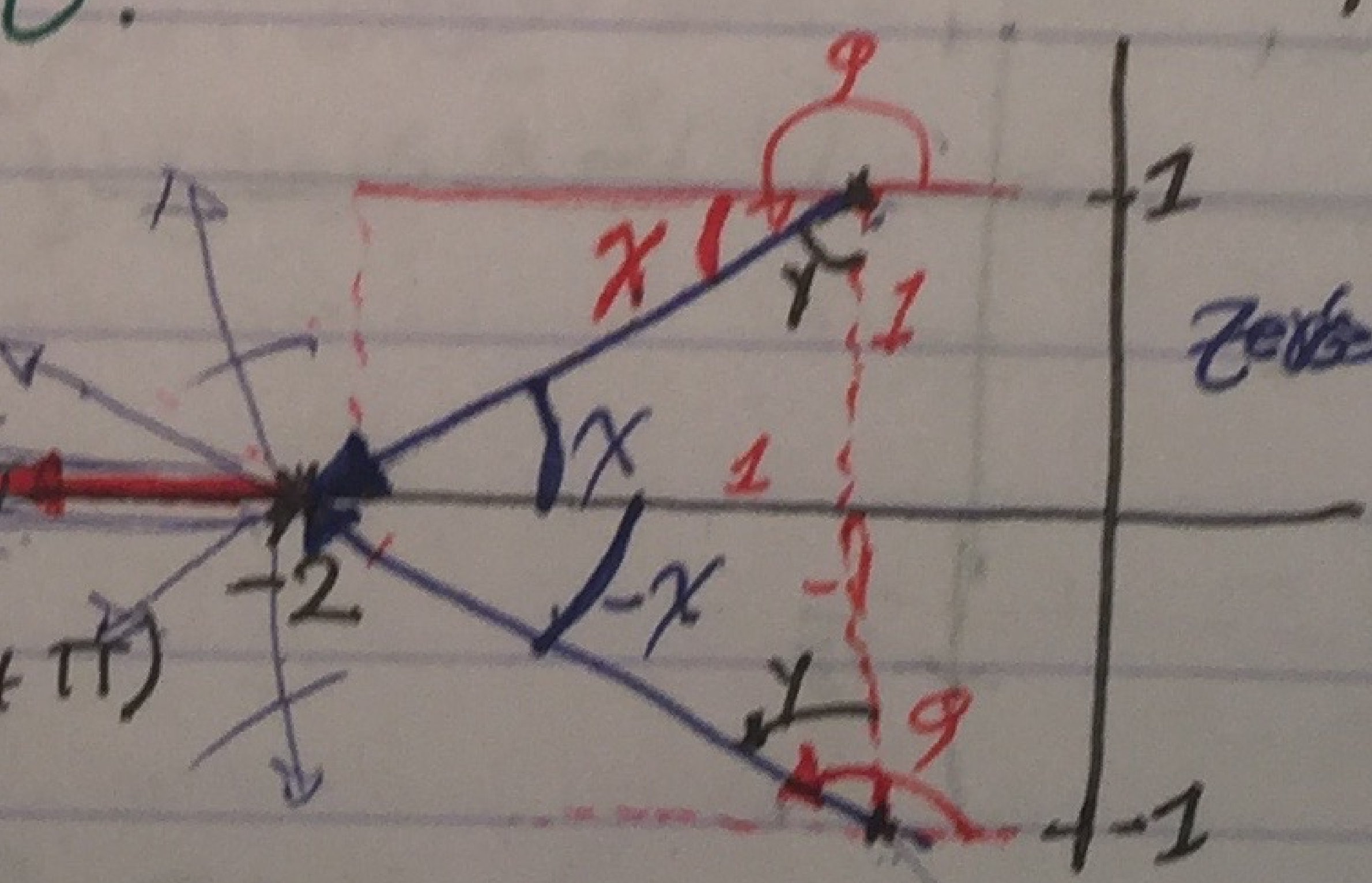


- ال complex لا نخدم (المستقبل) لأنهم دائماً يلغوا بعض
حيث لا نطلع عليهم ، نحيث نفقد ال poles و ال zeros
الموجودة على ال real axis

☒ Angles of departure.

$x = y = \frac{\pi}{4}$

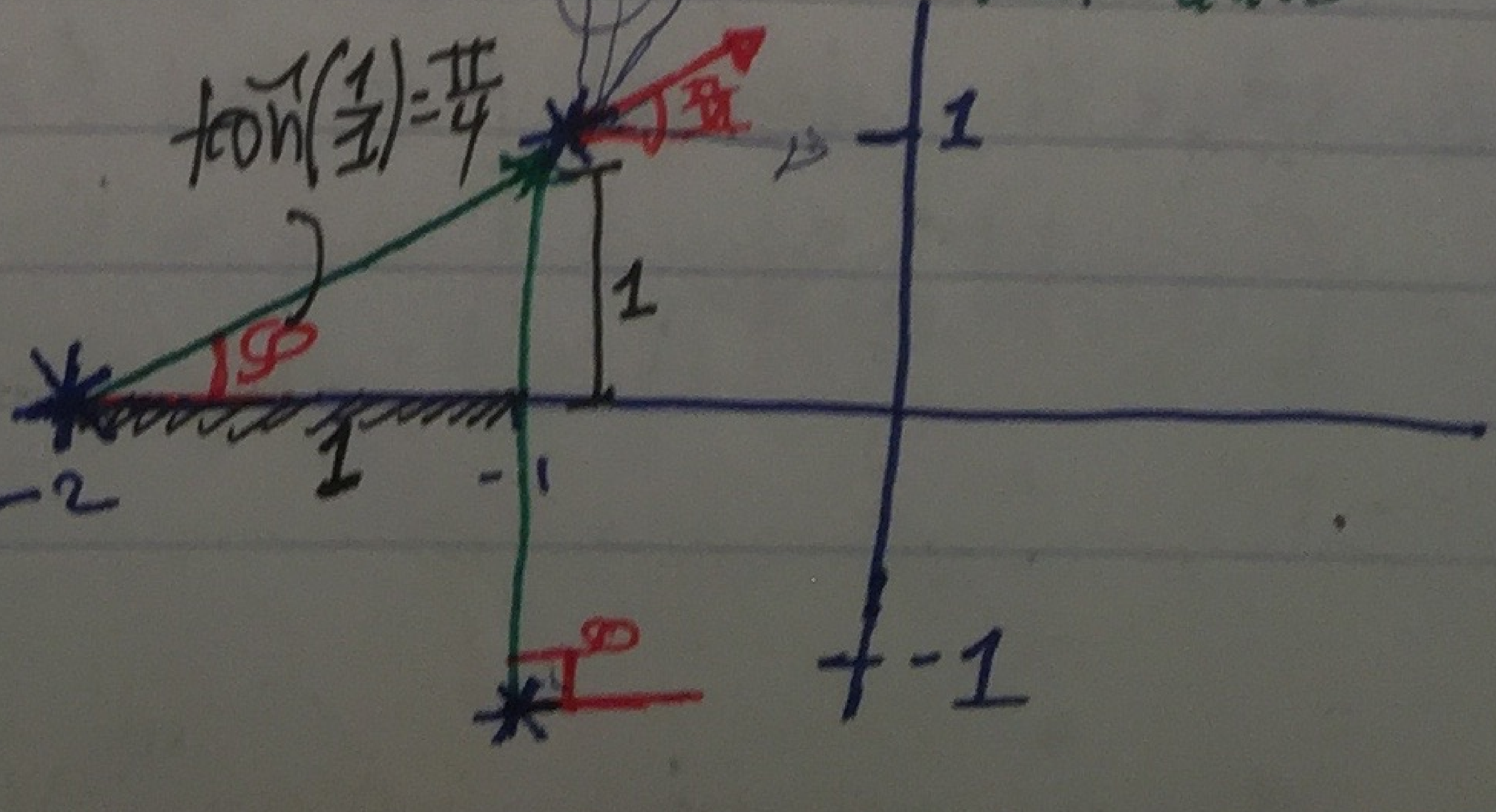
$$\varphi_{-2} = x - x + (2\nu + 1) = \pi$$
$$= (90 + y) + (180 + x) + (2\nu+1)\pi$$
$$= 90 + 45 + 180 + 45 + (2\nu + \pi)$$
$$= 135 + 225 + (2\nu + \pi)$$
$$= 135 - 135 + (2\nu + \pi) \overset{\nu=0}{=} \pi$$



الزوايا لل poles والzeros
ال Complex and Conju
تعطي زوايا معاكسة
بالنسبة لنقطة على
ال real axis

symmetry about the real axis

$$\varphi_{-1+j} = -\left(\frac{\pi}{4} + \frac{\pi}{2}\right) + (2\nu + 1)\pi$$
$$= -\frac{3\pi}{4} + \pi$$
$$= \frac{\pi}{4}$$



$\varphi_{-1-j} = \frac{-\pi}{4}$ by symmetry.

☒ Angles of arraival at infitity (the angle of the Assymptotic)

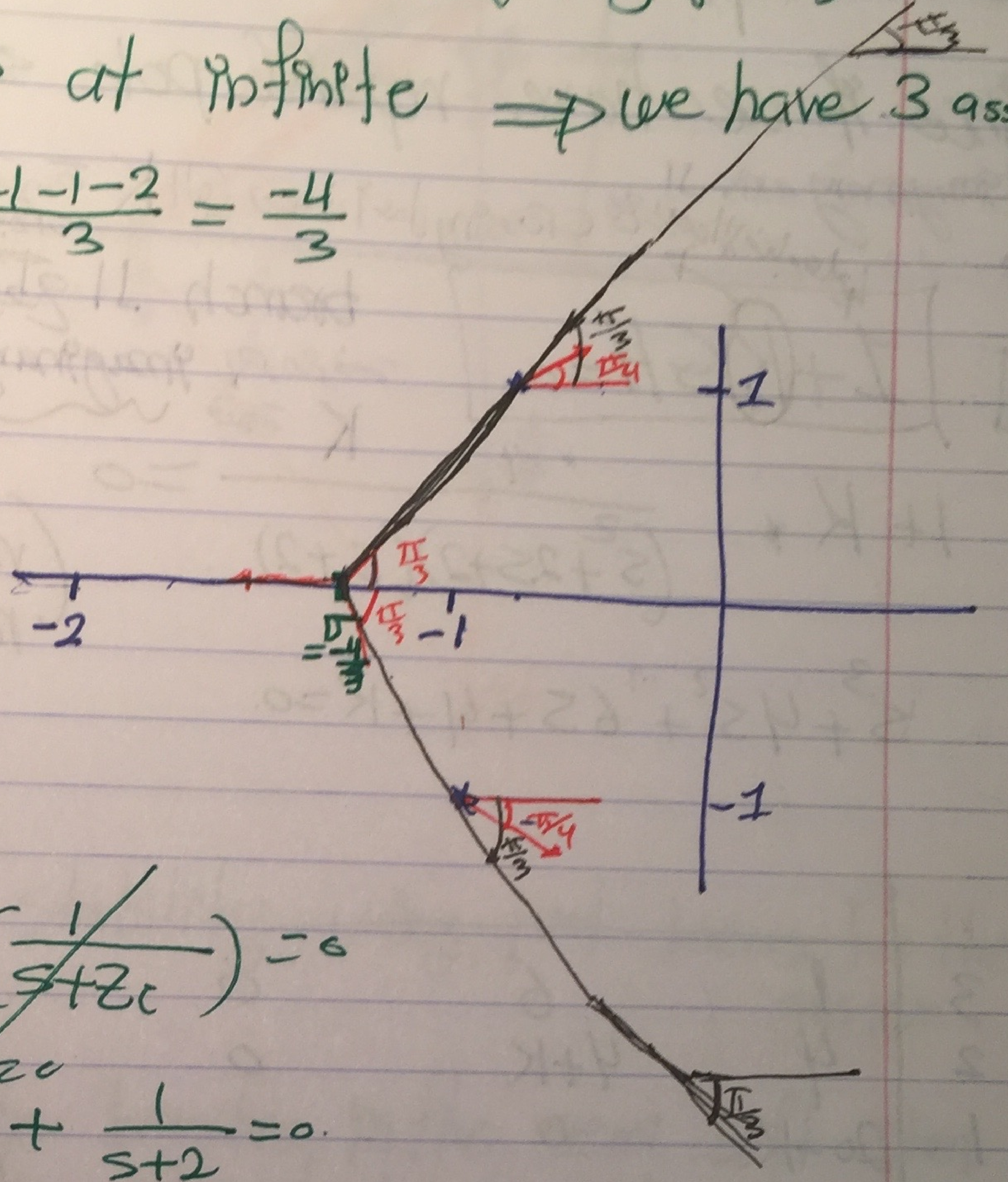
we have 3 zeros at infinite $\Rightarrow$ we have 3 assyptotic

Centroide $\sigma_c = \frac{-1-1-2}{3} = \frac{-4}{3}$

$\theta_0 = \frac{\pi}{3}$

$\theta_{-1} \overset{v=-1}{=} -\frac{\pi}{3}$

$\theta_1 = \pi$



☒ repeated poles.

$$\sum\left(\frac{1}{s+p_i}\right) \overset{\text{ناقص}}{-} \underset{=0}{\cancel{\sum\left(\frac{1}{s+z_i}\right)}} = 0$$

$$\frac{1}{(s+1+j)} + \frac{1}{(s+1-j)} + \frac{1}{s+2} = 0.$$

$$(s+1-j)(s+2) + (s+1+j)(s+2) + s^2+2s+2 = 0.$$

$$(s^2+3s+2) + (s^2+3s+2) + s^2+2s+2 = 0$$

$$3s^2+8s+6=0$$

$$\Delta = -8 < 0 \quad (\text{No solution})$$

as expected from geometry

- ممكن أن يلتقوا في نقطة Complex ولكن هنا لا يمكن أن يلتقوا 3
نقطة لأن إذا التقوا فإنه يجب أن تكون ال $\underbrace{5 = \text{order}}$
ولكن هنا ال 3 = order

2-Complex pole, 2-Conject
1-real pole.

So No repeated poles for Closed Loop.

since if we have repeated pole say how much sector

ال imaginary axis

مع الجزء الاستقرارية crossing اللي تجعلني أحصل K أيضا نستفيد لمعرفة قيمة cross

$\boxed{\times}$ $\boxed{1+\bigcirc\!\!\!\!K\,GH=0}$

وقلنا ب K بين يقطع ال branch ال imaginary axis بالمرة قيمة K

(row of zeros in routh Hwartiz table)

$$1+K*\frac{1}{(s^2+2s+2)(s+2)}=0$$

$$s^3+4s^2+6s+4+K=0$$

3	1	6	0
2	4	4+K	0
1	20-K	0	
0	4+K	0	

for K>0

$20-K=0 \Rightarrow K=20$ ✓

$4+K=0 \Rightarrow K=-4$ ✗

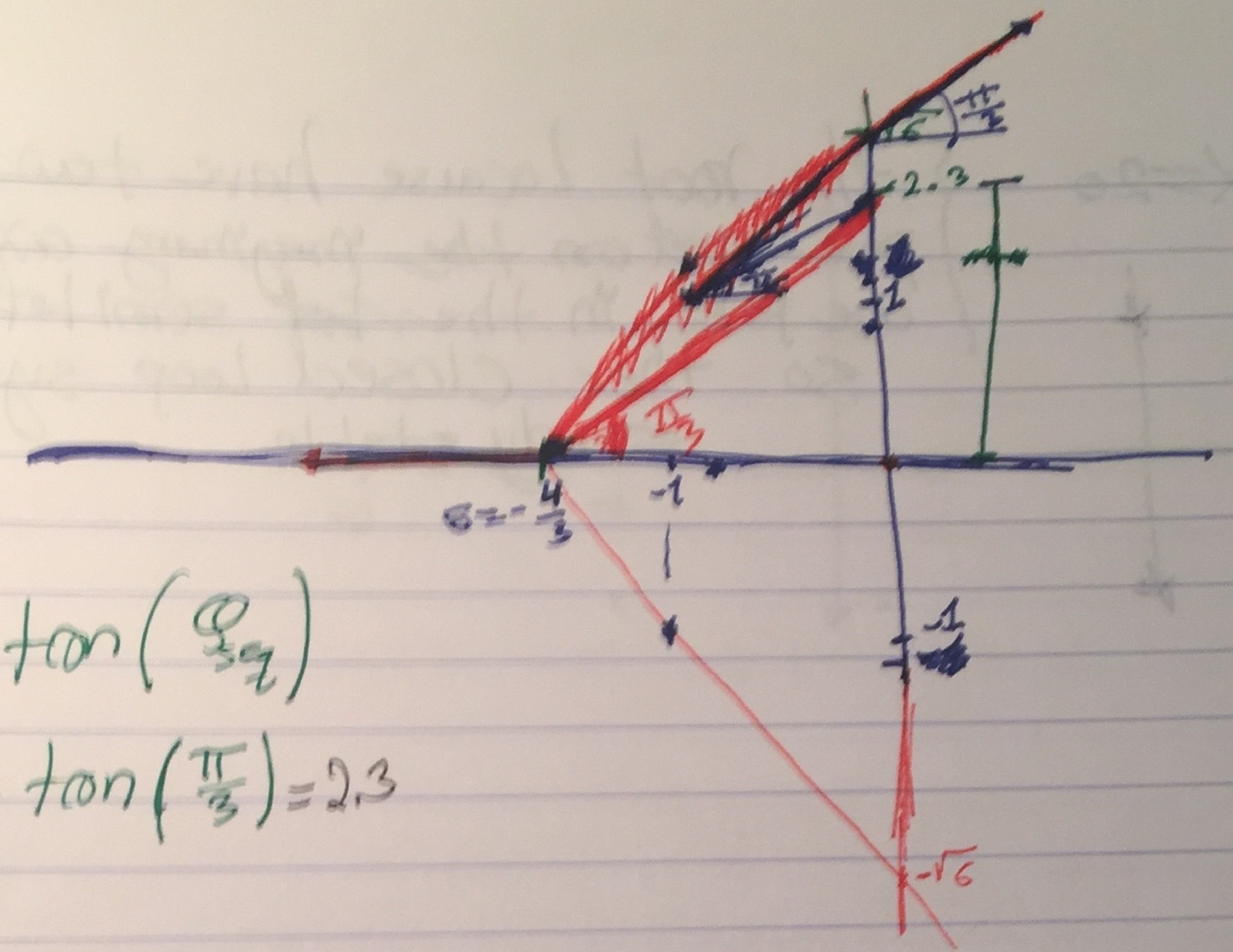
so we have.

$$A(s)=4s^2+(4+K)=4s^2+24$$

$$s_{1,2}=\mp\sqrt{6}$$

▨ Intersection of the asymptotic with the imaginary axis.

نفس النقطة نقطة ال symmetry

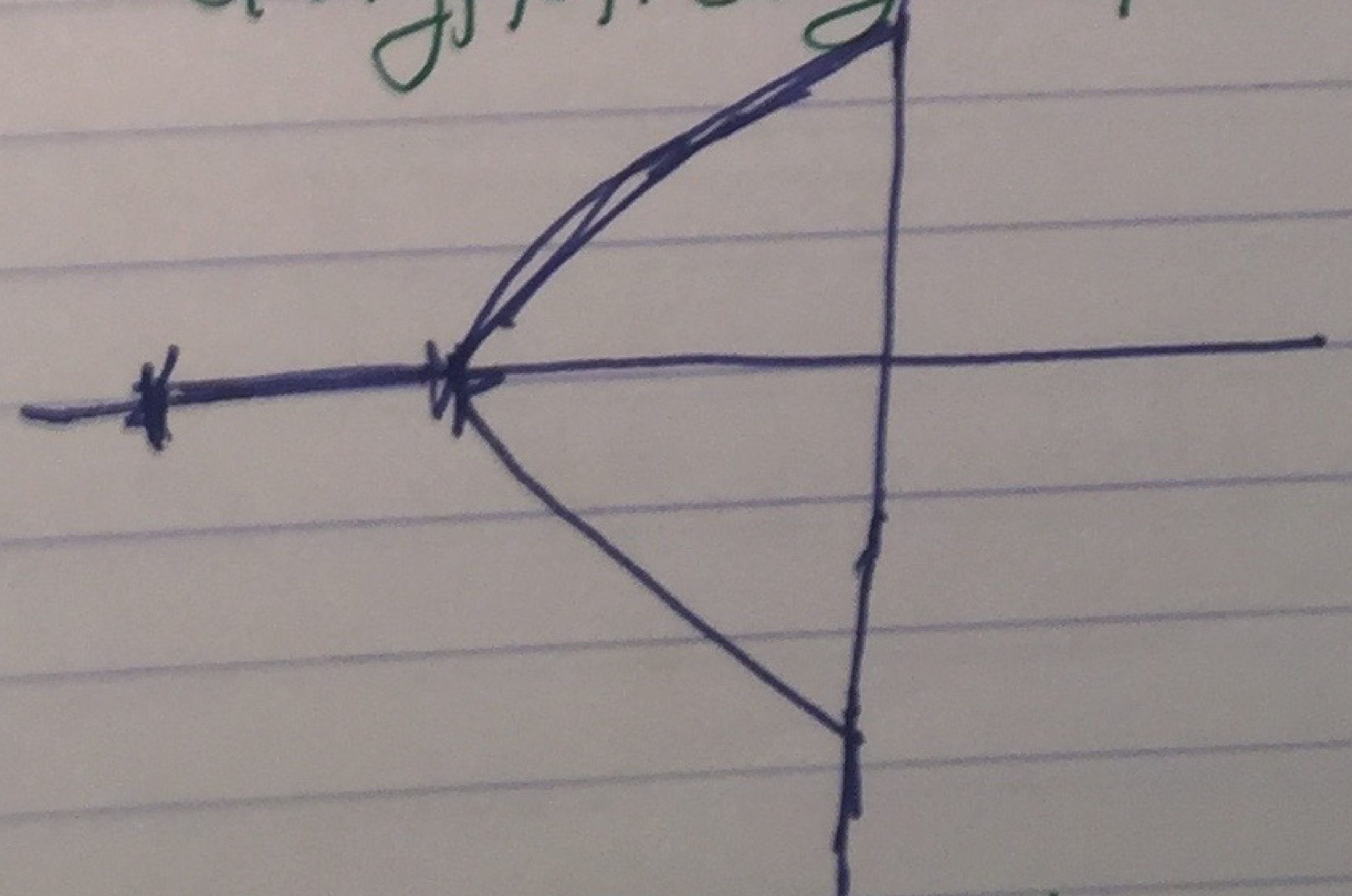


$$d = \sigma_c \tan\left(\varphi_{sq}\right)$$

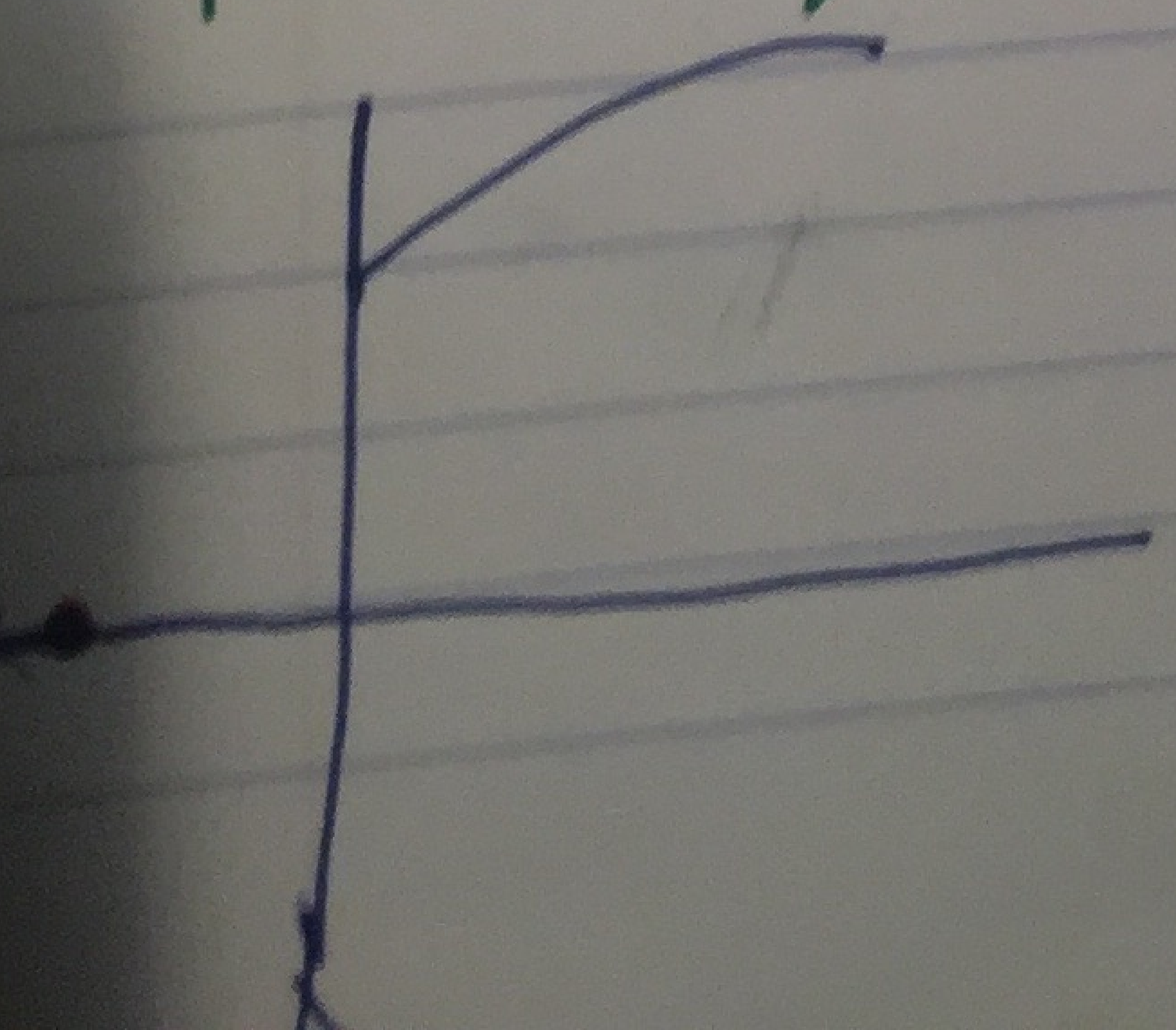
$$= \frac{4}{3} \tan\left(\frac{\pi}{3}\right) = 2.3$$

☒ Discuss the stability, using root Locus for the system.

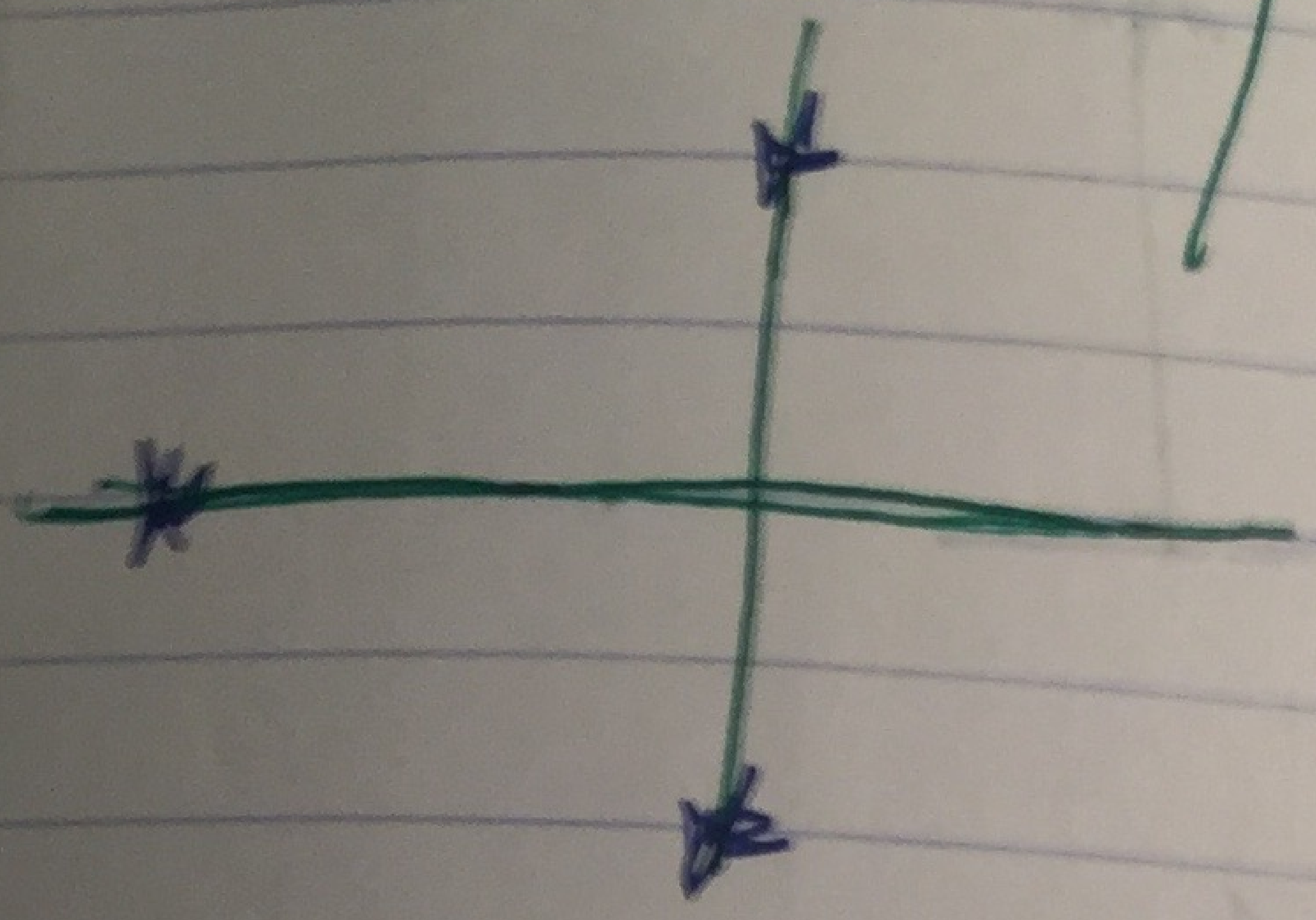
- For K < 20 { all the branches in the semi Left plan so the closed Loop system is assyptotically stable.



- For K > 20 { the root locus have two poles at the right half plan so the[illegible] have 2 sections from the branch at the ~~[illegible]~~ semi right plan, the closed loop system is unstable.



- For K=20

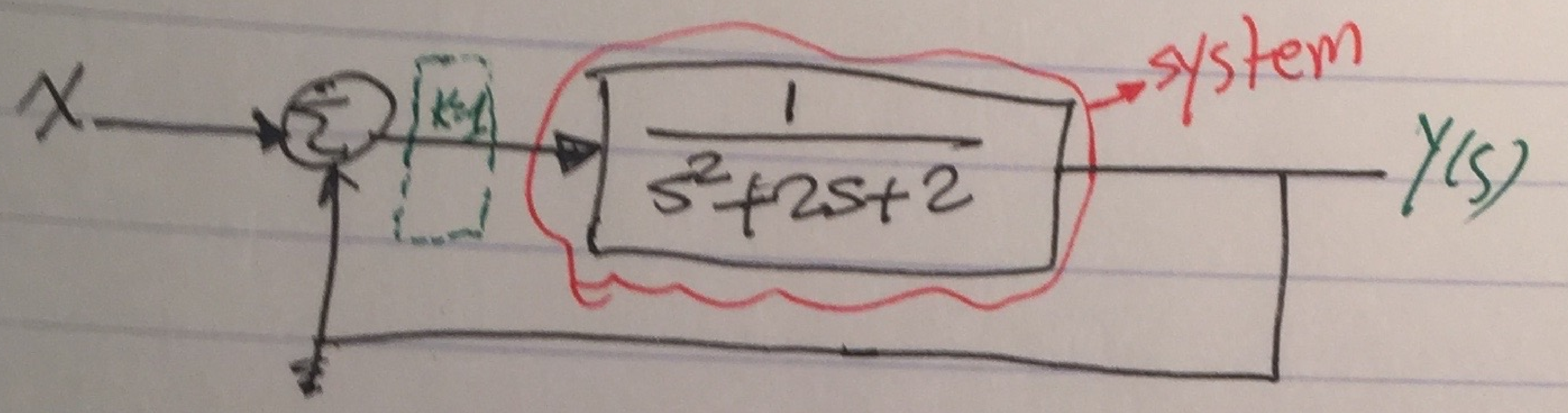


{ the root Locause have tow ymigynary pole ~~and~~ on the ymagynary axe and one pole in the ~~lef~~ semi left plane so the closed Loop system is simply stable.

EX

For this system Find.

1. The step response of the system.
2. plot the root Locause.
3. Discuss the possibilites for improving the SSE and the transfe of the system.
4. Improve the error by at Least 50% .



$\boxed{1}$ $H(s)=\dfrac{\frac{1}{s^2+2s+2}}{1+\frac{1}{s^2+2s+2}} = \dfrac{1}{s^2+2s+3} = \dfrac{Y(s)}{X(s)}$

$Y(s)=\dfrac{1}{s^2+2s+3} * X(s)$, the step response $X(s)=\frac{1}{s}$.

so we make Laplace inve

$Y(s)=\dfrac{1}{s^2+2s+3} * \dfrac{1}{s} = \dfrac{1}{s(s^2+2s+3)} = \underbrace{\dfrac{A}{s}+\dfrac{Bs+C}{s^2+2s+3}}$

-Note: the step response it is in time representation.

$A=\dfrac{1}{3}$

$1=A(s^2+2s+3)+(Bs+C)s$

$1=As^2+2As+3A+Bs^2+Cs$, $0=A+B \Rightarrow B=-\frac{1}{3}$

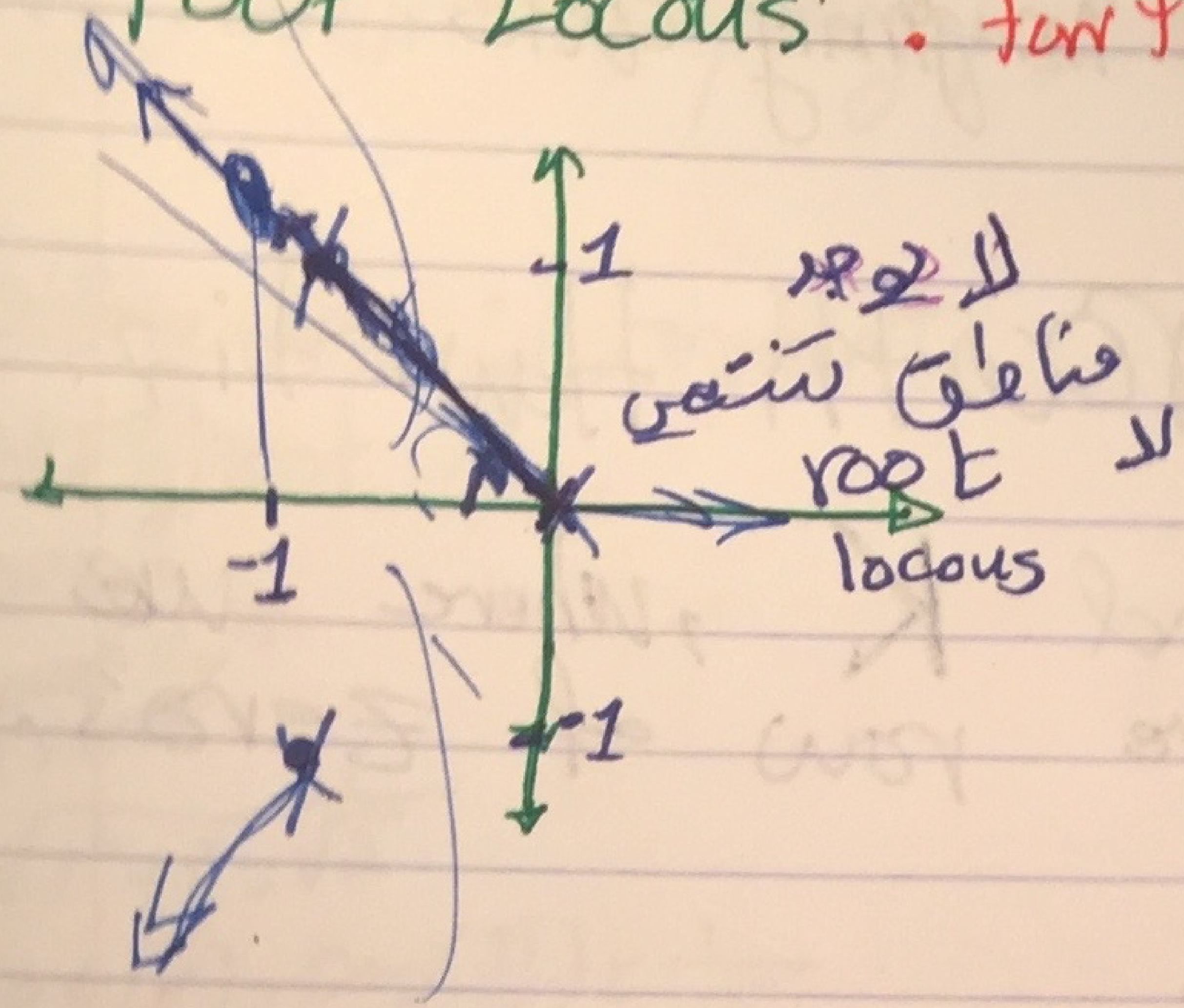
$0=2A+C \Rightarrow C=\dfrac{-2}{3}$

$=\dfrac{\frac{1}{3}}{s}+\dfrac{\frac{-1}{3}s}{s^2+2s+3}+\dfrac{\frac{-2}{3}}{s^2+2s+3}$

for [redacted] $G(s) \cdot H(s) = G(s) \cdot 1 = G(s) =$

open loop = A*B =

$\boxed{2}$ root Locous : for the system : the open Loop poles.



$$\frac{\overset{\boxed{k=1}}{1}}{(s^2+2s+2)} = \frac{\text{open Lopp system.}}{(s+j+1)(s-j+1)}$$

$$s = \frac{-2 \mp \sqrt{4-4 \cdot 2}}{2}$$

$$= -1 \mp j\frac{2}{2} = -1 \mp j$$

$s_1 = -1-j$

$s_2 = -1+j$.

$\boxed{1}$ Number of branches = Number of poles = [illegible] 2.

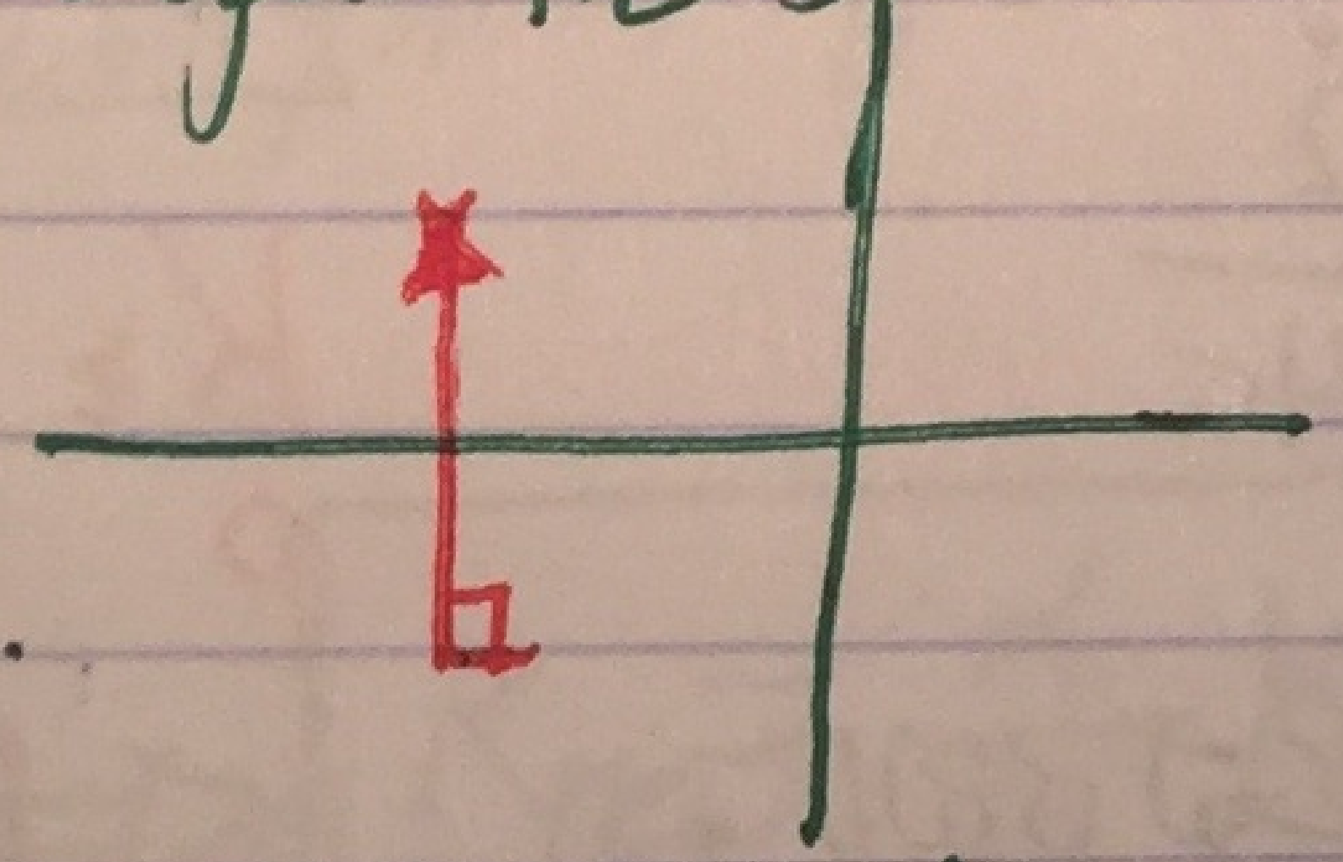
$\boxed{2}$ The root Locous is symmetric with respect to the real axis.

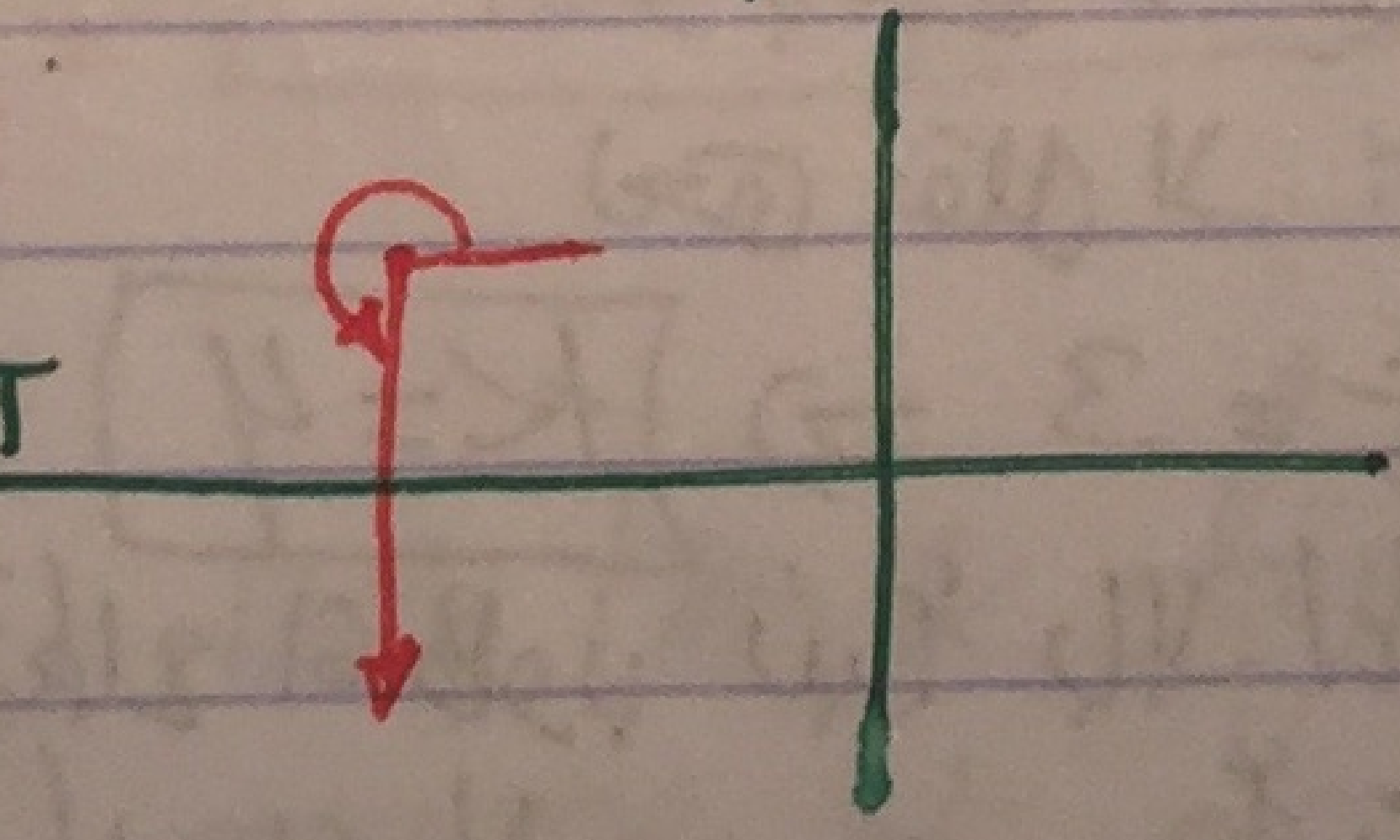
$\boxed{3}$ points of the real axis that belong to the locous Φ.

\- The angle of departion

$$\varphi_{-1+j} = 0 - \left(\frac{\pi}{2}\right) + (2l+1)\pi$$

$$= \pi - \frac{\pi}{2} = \frac{\pi}{2}.$$





$$\varphi_{-1-j} = 0 - \frac{3\pi}{2} + (2l+1)\pi$$

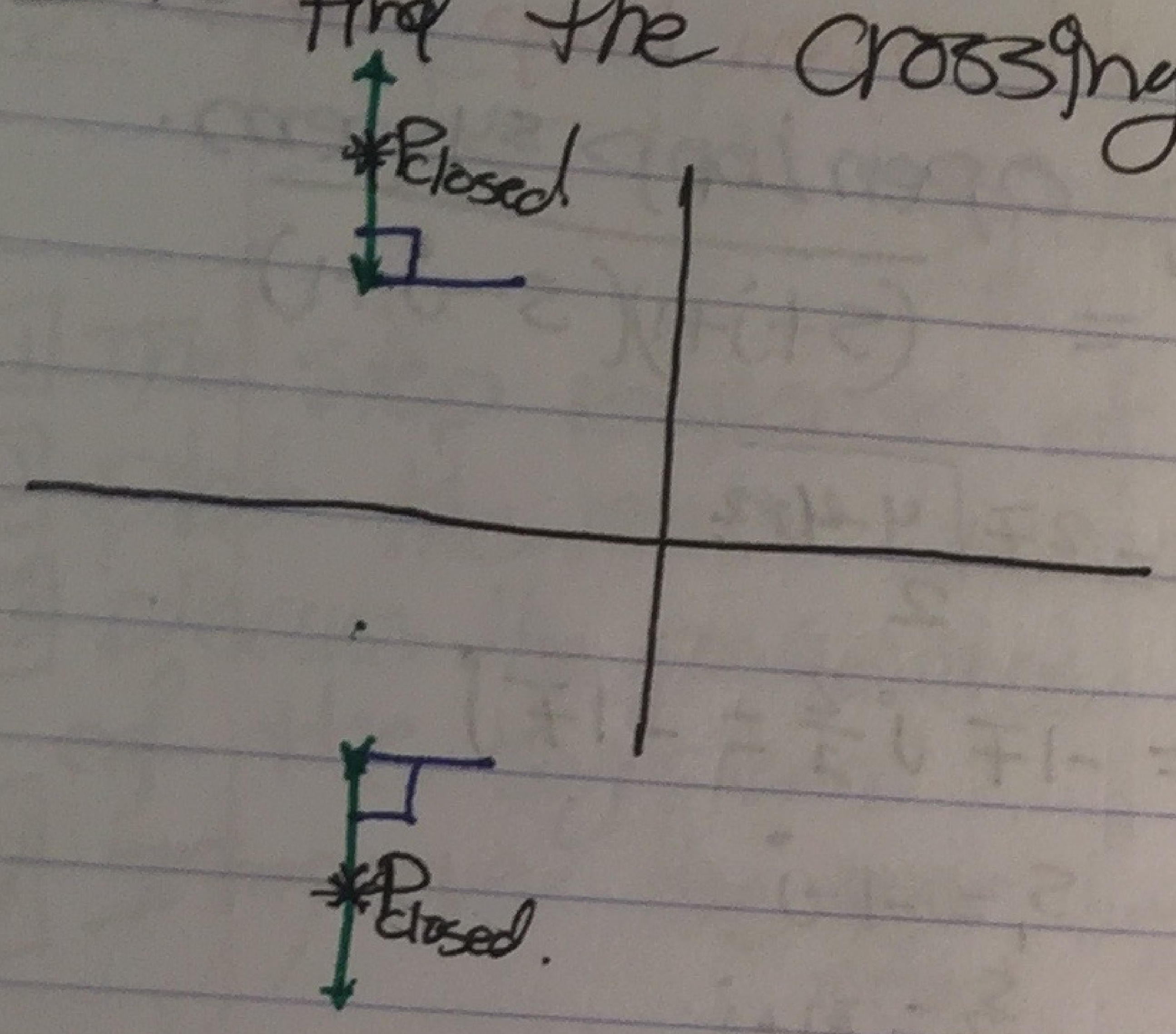
$$= \pi - \frac{3\pi}{2} = -\frac{\pi}{2}$$

\- repeated poles : From Graph No repeated poles

or by calculation $= 0$

$$\sum \frac{1}{s+P_i} - \sum \frac{1}{s+Z_i} = 0$$

- To find the crossing with the imaginary axis.



By routh Hawrtiz.

Find K, where we have row of zeros.

Because unity feedback the error ~~gain~~ function is the same for direct path function No poles in the origion from G so

$\boxed{3}$ zero order

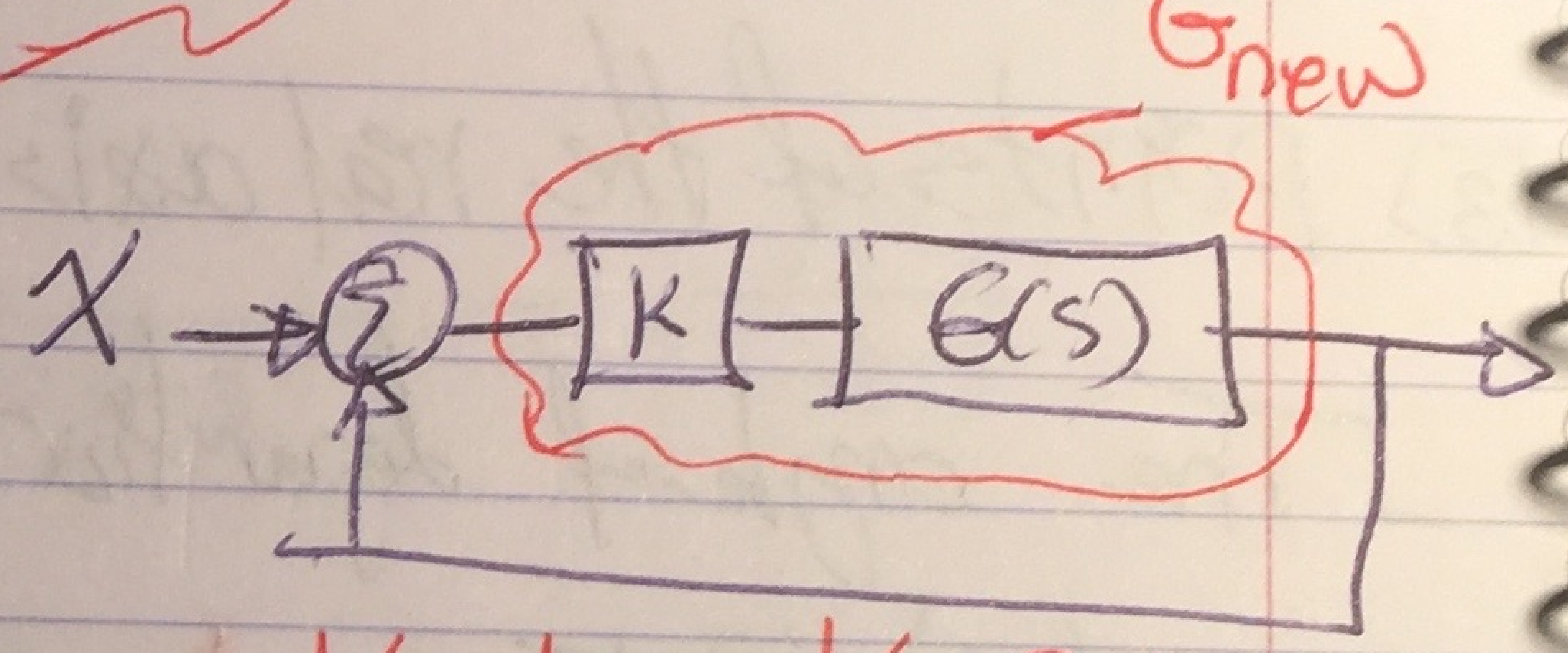
$$e_p = \frac{1}{1+K_p} , \quad K_p = \lim_{s\to 0} \frac{1}{s^2+2s+2} = \frac{1}{2}$$

→ G(s) (transfer function ليس ال).

$$e_p = \frac{1}{1+\frac{1}{2}} = \frac{2}{3}$$

عندما دخلنا ال K

$$e_p(K) = \frac{1}{1+\frac{K}{2}} = \frac{2}{2+K}$$



$$K_p = \lim_{s\to 0} K * G(s)$$

$$= \lim_{s\to 0} \frac{K}{s^2+2s+2} = \frac{K}{2}$$

وبالتالي لتحسين ال error يجب أن نزيد قيمة ال K حتى نقلل ال error، مثلاً إذا أردنا أن نجعله $\frac{1}{3}$

$$\frac{1}{1+\frac{K}{2}} = \frac{1}{3} \Rightarrow 1+\frac{K}{2} = 3 \Rightarrow \boxed{K=4}$$

* إذا كان بالإمكان أن نطلع المطلوب نبقى بال static contrallel

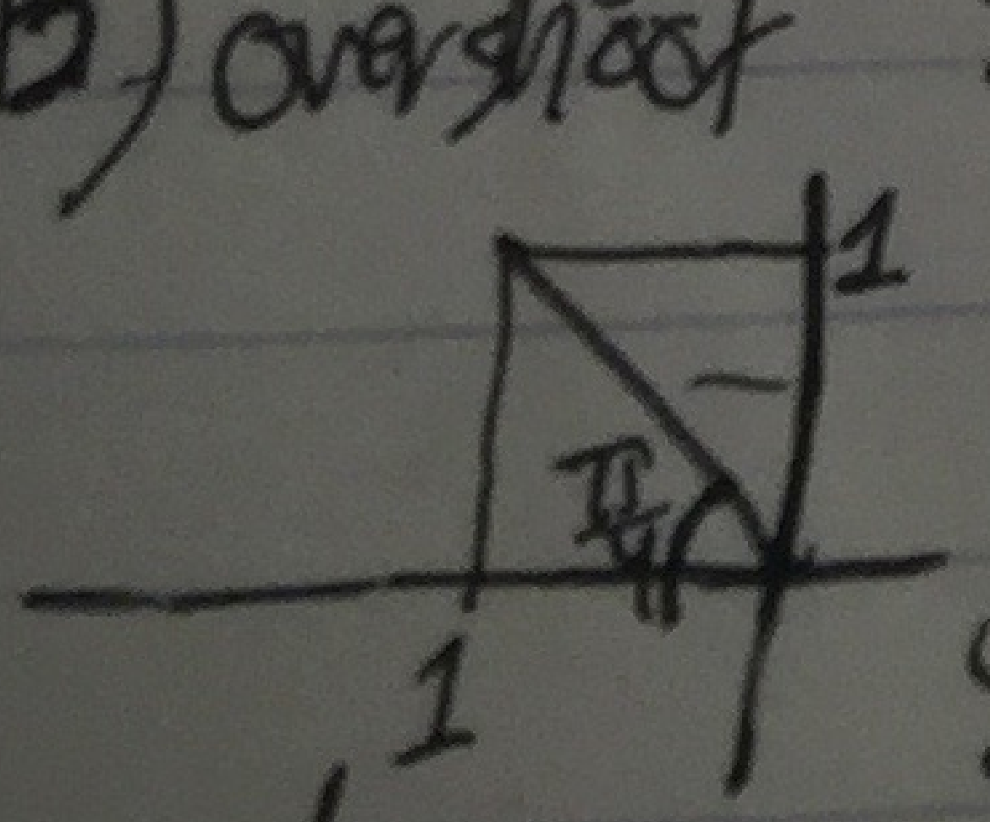
أما إذا ما قدر يطلع المطلوب يجب أن ندخل ال Dynamic

- هنا عشان $K \to \infty$ فإنه $e_p = 0$. Dynamic controller. لازم أضيف

في ال transient نناقش ال overshoot, seltling time, damping freq. ↓ Dynamic controller

a) seltling time: Because the real part constant, so we can't chang it

b) overshoot: we can chang it, but to limit., since



} minimume over shoot.

(Natural frequency).

c) damping frequency: (imaginary part for pole): from $(1 \to \infty)$

when $K\uparrow \Rightarrow$ oscillation frequency $\uparrow \Rightarrow$ error $\downarrow$ $\left(\omega_n = \sqrt{2} \to \infty\right)$ min max

يجب أن نضيف ال Dynamic controller.

$K\uparrow$, osillation frequency $\uparrow$, Error $\downarrow$

Damping Controller:

Lag compinsator ← لا يصبح صفر ال error

PI controller ← ال error يصبح صفر

settling time : Constant

overshoot :

يجب ان نعرف ال minimum overshoot

$= \sqrt{1^2+1^2} = \frac{1}{\sqrt{2}}$.

$= \cos(\xi) = \cos(\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$.

- تحسين ال error باستخدام ال lag comp. يعني

$$\frac{K_{new}}{K_{old}} = \frac{\alpha}{\beta}.$$

$$e_{new} < e_{old} \Rightarrow \underset{\text{static gain}}{\boxed{K_s}} > K_{s,old}.$$

$$G_{lag} = \frac{s+\alpha}{s+\beta}, \ |\alpha| > |\beta|$$

$$K_{s,old} = \lim_{s \to 0} s^i G(s), \quad i: \text{system type}$$

$$K_{old} = K \frac{\Pi Z_i}{\Pi P_i}$$

$$K_{new} = K \frac{\Pi Z_i}{\Pi P_i} * \frac{\alpha}{\beta} \Rightarrow \boxed{K_{new} = \frac{\alpha}{\beta} K_{old}} \rightarrow$$

$\alpha > B$ وبالتالي عندنا تحسين على ال error.

[4] $\frac{K_{new}}{K_{old}} = \frac{\alpha}{\beta}$, we have degree of freedom so we assume α or β.

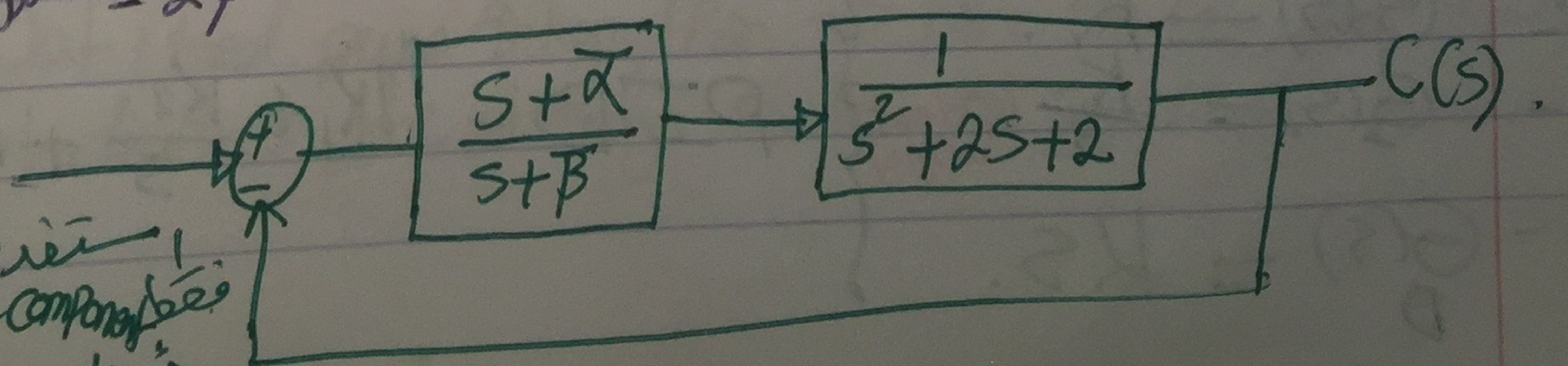
في العادة نختار ال β لأنه نفسه ال Zero بيروح باتجاه اليسار، ونحافظ على ال order لل system.

So $\bar{\alpha} = \bar{\beta} \left(\frac{K_{new}}{K_{old}} \right)$

نفرض قيمة $= 2\bar{\beta}$

$error_{new} = \frac{1}{2} error_{old} \Rightarrow K_{new} = 2K_{old}$.

So



استخدم اذا استخدمنا فقط (R,L,C) passive components ووفرنا ال Amplifier.