8.2 Trigonometric Integral:

* Integral of the form $\int \sin^m x \cos^n x \, dx$, where n, m are non negative integer

We have 3-cases

Case 1:- If m is odd, we write $m = 2K+1$, and use $\sin^2 x = 1 - \cos^2 x$

Case 2:- If ~~##~~ n is odd, we write $n = 2K+1$, and use $\cos^2 x = 1 - \sin^2 x$

Case 3: If Both n, m are even, we use
$$\cos^2(x) = \frac{1}{2}\left[1 + \cos(2x)\right], \quad \sin^2(x) = \frac{1}{2}\left[1 - \cos(2x)\right]$$

Exp1 Evaluate the following integral:-

$\boxed{1}$ $\int \sin x \cos^4 x \, dx$

$U = \cos x$

$dU = -\sin x \, dx$

$$\int -U^4 \, dU = \frac{-U^5}{5} + C$$
$$= \frac{-1}{5} \cos^5 x + C$$

$\boxed{2}$ $\int \sin^3 x \cos^4 x \, dx$

$\int \sin x \sin^2 x \cos^4 x \, dx$

$U = \cos x$

$dU = -\sin x \, dx$

$\int \sin x \left[1 - \cos^2 x\right] \cos^4 x \, dx$

$-\int \left[1 - U^2\right] U^4 \, dU$

$= \int U^6 - U^4 \, dU = \frac{U^7}{7} - \frac{U^5}{5} + C$

$= \frac{1}{7} \cos^7 x - \frac{1}{5} \cos^5 x + C$

$\boxed{3}$ $\int \sin^6 x \cos^5 x \, dx$

$U = \sin x$

$dU = \cos x \, dx$

$\int \sin^6 x \left[\cos^2 x\right]^2 \cos x \, dx$

$\int \sin^6 x \left[1 - \sin^2 x\right]^2 \cos x \, dx$

$\int U^6 \left[1 - U^2\right]^2 dU$

$\int U^6 \left[1 - 2U^2 + U^4\right] dU = \frac{U^7}{7} - 2\frac{U^9}{9} + \frac{U^{11}}{11} + C$

$= \frac{1}{7} \sin^7(x) - \frac{2}{9} \sin^9(x) + \frac{1}{11} \sin^{11}(x) + C$

$\boxed{4}$ $\int \sin^3 x \cos^3 x \, dx$

$$\int \sin^3 x \cos^2 x \cos x \, dx$$

$$\int \sin^3 x \left[1 - \sin^2 x\right] \cos x \, dx \qquad \begin{aligned} u &= \sin x \\ du &= \cos x \, dx \end{aligned}$$

$$\int u^3 \left[1 - u^2\right] du = \frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + C$$

$\boxed{5}$ $\int \sin^4 x \cos^2 x \, dx$

$$\int \sin^2 x \sin^2 x \cos^2 x \, dx$$

$$\int \sin^2 x \left[\frac{1}{2} \sin(2x)\right]^2 dx$$

$$\frac{1}{4} \int \sin^2 x \sin^2(2x) \, dx$$

$$\frac{1}{4} \cdot \frac{1}{2} \int \left[1 - \cos(2x)\right] \sin^2(2x) \, dx$$

$$\frac{1}{8} \int \sin^2 x \, dx - \frac{1}{8} \int \cos(2x) \sin^2(2x) \, dx \qquad \begin{aligned} u &= \sin(2x) \\ du &= 2 \cos(2x) \, dx \end{aligned}$$

$$\frac{1}{8} \cdot \frac{1}{2} \int 1 - \cos(4x) \, dx - \frac{1}{8} \int \frac{u^2}{2} \, du$$

$$\frac{1}{16} x - \frac{1}{64} \sin(4x) - \frac{1}{48} \sin^3(2x) + C$$

* Eliminating Square Root

Exp: Evaluate the following integral:-

$\boxed{1}$ $$\int_0^{\pi/2} x\sqrt{1-\cos(2x)}\,dx$$

$$\int_0^{\pi/2} x\sqrt{2\sin^2 x}\,dx$$

$$\sqrt{2}\int_0^{\pi/2} x\,|\sin x|\,dx$$

$u = x$ $\quad dv = \sin x\,dx$

$dx = dx$ $\quad v = -\cos x$

Derivative		Integral
		$\sin x$
x	$+$	$-\cos x$
1	$-$	$-\sin x$
0		

$$\sqrt{2}\left[-x\cos x - \int -\cos x\,dx\right]$$

$$\sqrt{2}\left[-x\cos x + \sin x\right]_0^{\pi/2} = \sqrt{2}$$

$\boxed{2}$ $$\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1-\cos x}}\,dx$$

$$\sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$\sin^2\left(\frac{x}{2}\right) = \frac{1-\cos(x)}{2}$$

$$\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{2\sin^2\left(\frac{x}{2}\right)}}\,dx$$

$$\sin(2x) = 2\sin x\cos x$$

$$\sin(x) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$

$$\frac{1}{\sqrt{2}}\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\left|\sin\left(\frac{x}{2}\right)\right|}\,dx$$

$U = \cos\left(\frac{x}{2}\right)$

$dU = \frac{-1}{2}\sin\left(\frac{x}{2}\right)dx$

$$\frac{1}{\sqrt{2}}\int_{\pi/3}^{\pi/2} 4\sin\left(\frac{x}{2}\right)\cos^2\left(\frac{x}{2}\right)dx = 2\sqrt{2}\int_{\frac{\sqrt{3}}{2}}^{\frac{1}{\sqrt{2}}} -2U^2\,dU = -4\sqrt{2}\,\frac{U^3}{3}\Bigg]_{\frac{\sqrt{3}}{2}}^{\frac{1}{\sqrt{2}}}$$

$$= \frac{-2}{3} + \frac{\sqrt{3}}{\sqrt{2}}$$

$$\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1-\cos x}}\, dx$$

$$\int_{\pi/3}^{\pi/2} \frac{\sin^2 x \sqrt{1+\cos x}}{\sqrt{1+\cos x} \cdot \sqrt{1-\cos x}}\, dx$$

$$\int_{\pi/3}^{\pi/2} \frac{\sin^2 x \sqrt{1+\cos x}}{|\sin x|}\, dx = \int_{\pi/3}^{\pi/2} \sin x \sqrt{1+\cos x}\, dx$$

$$= \int_{3/2}^{1} -\sqrt{U}\, dU$$

$$= \int_{1}^{3/2} U^{1/2}\, dU = \frac{2}{3} U^{3/2} \Big]_{1}^{3/2}$$

$$= \frac{\sqrt{3}}{\sqrt{2}} - \frac{2}{3}$$

* Integral of power tan x sec x

Evaluate the following integral:-

$\boxed{1}$ $\int \sec^6 x \, dx$

$\int \sec^4 x \sec^2 x \, dx$

$U = \tan x$

$du = \sec^2 x \, dx$

$\int [\tan^2 x + 1]^2 \sec^2 x \, dx$

$\int [U^2 + 1]^2 \, dU$

$\int U^4 + 2U^2 + 1 \, dU = \frac{U^5}{5} + \frac{2U^3}{3} + U + C$

$= \frac{1}{5} \tan^5(x) + \frac{2}{3} \tan^3(x) + \tan x + C$

$\boxed{2}$ $\int \tan^4(x) \, dx$

$\int \tan^2 x \tan^2 x \, dx$

$\int [\sec^2 x - 1] \tan^2 x \, dx = \int \sec^2 x \tan^2 x \, dx - \int \tan^2 x \, dx$

$U = \tan x$

$dU = \sec^2 x \, dx$

$= \int U^2 \, dU - \int \sec^2 x - 1 \, dx$

$= \frac{U^3}{3} - \tan x + x + C$

$= \frac{1}{3} \tan^3 x - \tan x + x + C$

$\boxed{3}$ $\int 4\tan^3 x \, dx = 4\int (\sec^2 x - 1)\tan x \, dx$

$= 4\int \sec^2 x \tan x \, dx - 4\int \tan x \, dx$

$u = \tan x$
$du = \sec^2 x \, dx$

$= 4 \frac{u^2}{2} - 4 \ln|\sec x| + C$

$= 2\tan^2 x - 2\ln|\tan^2 x + 1| + C$

$\boxed{4}$ $\int \sec x \tan^2 x \, dx = \int \sec x \tan x \tan x \, dx$

$u = \tan x$ $\quad dv = \tan x \sec x \, dx$

$du = \sec^2 x \, dx$ $\quad v = \sec x$

$= \tan x \sec x - \int \sec x \sec^2 x \, dx$

$= \tan x \sec x - \int \sec x [\tan^2 x + 1] \, dx$

$= \tan x \sec x - \int \sec x \tan^2 x \, dx - \int \sec x \, dx$

$\int \sec x \tan^2 x \, dx = \tan x \sec x - \int \sec x \tan^2 x \, dx - \ln|\sec x + \tan x|$

$\int \sec x \tan^2 x \, dx = \frac{1}{2}\tan x \sec x - \frac{1}{2}\ln|\sec x + \tan x| + C$

$\boxed{5}$ $\int \sec^3 x \tan^3 x \, dx$

$\int \sec^2 x \tan^2 x \sec x \tan x \, dx$

$\int \sec^2 x \left[\sec^2 x - 1\right] \sec x \tan x \, dx$

$U = \sec x$

$dU = \sec x \tan x \, dx$

$\int U^2 \left[U^2 - 1\right] dU$

$\int U^4 - U^2 \, dU = \frac{U^5}{5} - \frac{U^3}{3} + C$

$= \frac{1}{5}\sec^5 x - \frac{1}{3}\sec^3 x + C$

$\boxed{6}$ $\int \sec^4 x \tan^4 x \, dx$

$= \int \sec^2 x \sec^2 x \tan^4 x \, dx$

$U = \tan x$

$dU = \sec^2 x \, dx$

$\int \left[\tan^2 x + 1\right] \tan^4 x \sec^2 x \, dx$

$\int \left[U^2 + 1\right] U^4 \, dU = \int U^4 + U^6 \, dU$

$= \frac{U^5}{5} + \frac{U^7}{7} + C$

$= \frac{1}{5}\tan^5(x) + \frac{1}{7}\tan^7(x) + C$

Q63

$\boxed{4}$. $\displaystyle\int \frac{\sec^3 x}{\tan x}\, dx$

$$\int \frac{\sec^2 x \ \sec x}{\tan x}\, dx$$

$$\int \frac{(1+\tan^2 x)\ \sec x}{\tan x}\, dx$$

$$\int (\cot x + \tan x) \sec x \ dx$$

$$\int \cot x \ \sec x \, dx + \int \tan x \ \sec x \ dx$$

$$\int \frac{\cos x}{\sin x} \cdot \frac{1}{\cos x}\, dx + \sec x + C$$

$$\int \csc x \ dx + \sec x + C$$

$$-\ln |\csc x + \cot x| + \sec x + C.$$

* Integral of product of sin (ax) and cos (bx)

– We use the following formula:–

$$\sin(ax)\cos(bx) = \frac{1}{2}\left[\sin(a-b)x + \sin(a+b)x\right]$$

$$\sin(ax)\sin(bx) = \frac{1}{2}\left[\cos(a-b)x - \cos(a+b)x\right]$$

$$\cos(ax)\cos(bx) = \frac{1}{2}\left[\cos(a-b)x + \cos(a+b)x\right]$$

Evaluate the following integral:

$\boxed{1}$ $\int \sin(2x)\cos(3x)\,dx$

$$\int \frac{1}{2}\left[\sin(-x) + \sin(5x)\right] dx$$

$$\frac{1}{2}\int \sin(-x)\,dx + \frac{1}{2}\int \sin(5x)\,dx$$

$$\frac{-1}{2}\int \sin(x)\,dx + \frac{1}{2}\int \sin(5x)\,dx$$

$$\frac{-1}{2} - \cos x + \frac{1}{2}\,\frac{-\cos(5x)}{5} + C$$

$$\frac{1}{2}\cos x - \frac{1}{10}\cos(5x) + C$$

Q55

$\boxed{2}$ $\int \cos(3x) \cos(4x)\, dx$

$\int \frac{1}{2}\left[\cos(-x) + \cos(7x)\right] dx$

$\frac{1}{2}\int \cos(x)\, dx + \frac{1}{2}\int \cos(7x)\, dx$

$\frac{1}{2}\sin x + \frac{1}{2}\frac{\sin(7x)}{7} + C$

Q61

$\boxed{3}$ $\int \sin(\theta) \cos(\theta) \cos(3\theta)\, d\theta$

$\int \frac{1}{2}\sin(2\theta) \cos(3\theta)\, d\theta$

$\frac{1}{2}\frac{1}{2}\int \left[\sin(-\theta) + \sin(5\theta)\right] d\theta$

$\frac{1}{4} - \int \sin(\theta) + \frac{1}{4}\int \sin(5\theta)\, d\theta$

$-\frac{1}{4} - \cos\theta + \frac{1}{4}\frac{-\cos(5\theta)}{5} + C$

$\frac{1}{4}\cos\theta - \frac{1}{20}\cos(5\theta) + C$

$\boxed{33}$ $\int \sec^2 x \tan x \, dx$

$U = \tan x$
$dU = \sec^2 x \, dx$

$\int U \, dU = \frac{U^2}{2} + C$

$= \frac{1}{2} \tan^2 x + C$

$\boxed{45}$ $\int 4 \tan^3 x \, dx$

$4 \int \tan x \tan^2 x \, dx$

$4 \int \tan x \left[\sec^2 x - 1 \right] dx = 4 \int \tan x \sec^2 x \, dx - 4 \int \tan x \, dx$

$U = \tan x$
$dU = \sec^2 x \, dx$

$= 4 \int U \, dU - 4 \ln |\sec x| + C$

$= 4 \frac{\tan^2 x}{2} - 4 \ln |\sec x| + C$

$= 2 \tan^2 x - 2 \ln |\tan^2 x + 1| + C$

$\boxed{67}$ $\int x \sin^2 x \, dx = \int x \left(\frac{1 - \cos 2x}{2} \right) dx$

$= \frac{1}{2} \int x \, dx - \frac{1}{2} \int x \cos(2x) \, dx$

$U = x$ $\quad dV = \cos(2x) \, dx$
$dU = dx$ $\quad V = \frac{\sin(2x)}{2}$

$= \frac{1}{2} \cdot \frac{x^2}{2} - \frac{1}{2} \cdot \frac{1}{2} x \sin(2x) + \frac{1}{2} \int \frac{\sin(2x)}{2} dx$

$= \frac{1}{4} x^2 - \frac{1}{4} x \sin(2x) - \frac{1}{8} \cos(2x).$