Exercises :

Q7: let X, Y have a bivariate normal distribution with parameters $\mu_1 = 3$, $\mu_2 = 1$, $\sigma_1^2 = 16$, $\sigma_2^2 = 25$ and $\rho = \frac{3}{5}$. determine the following prob.

a. $Pr(3 < Y < 8)$. $Y \sim N(1, 5^2)$

$Z = \frac{Y - \mu_2}{\sigma_2} \Rightarrow Pr\left(\frac{2}{5} < Z < \frac{7}{5}\right) = Pr(0.4 < Z < 1.4)$

$= \phi(1.4) - \phi(0.4)$

$= 0.919 - 0.655$

$= 0.264$

b. $Pr(3 < Y < 8 \mid X = 7)$

$(Y|X) \sim N\left(\mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x - \mu_1), \ \sigma_2^2(1 - \rho^2)\right)$

$(Y|X) \sim N\left(1 + \frac{3}{\cancel{5}}\left(\frac{\cancel{5}}{\cancel{4}}\right)(\cancel{7-3}^{\,4}), \ 25\left(1 - \left(\frac{3}{5}\right)^2\right)\right)$

$25\left(\frac{25}{25} - \frac{9}{25}\right)$

$(Y|X) \sim N(4, 4^2)$

Now $\rightarrow$ $Z = \frac{Y - 4}{4} \Rightarrow Pr\left(-\frac{1}{4} < Z < 1\right)$ $Z \sim N(0,1)$.

$= Pr(-0.25 < Z < 1)$

$= \phi(1) - (1 - \phi(0.25))$

$= 0.841 - (1 - 0.599)$

$= 0.44$

c. $Pr(-3 < X < 3)$.

$X \sim (3, 4^2)$

$Z = \frac{X-3}{4} \Rightarrow Pr\left(\frac{-6}{4} < Z < 0\right) = Pr(-1.5 < Z < 0)$

$= \phi(0) - (1 - \phi(1.5))$

$= 0.5 - (1 - 0.933)$

$= 0.433$.

d. $Pr(-3 < x < 3 \mid y = 4)$.

$x|y \sim N\left(\mu_1 + \rho \frac{\sigma_1}{\sigma_2}(y - \mu_2),\ \sigma_1^2(1 - \rho^2)\right)$

$x|y \sim N\left(3 + \frac{3}{5}\left(\frac{4}{5}\right)(-4-1),\ 16\left(1 - \frac{9}{25}\right)\right)$

$10.24 = 3.2^2$

$x|y \sim N(0.6,\ 3.2^2)$

So $Z = \frac{X - 0.6}{3.2} \Rightarrow Pr(-1.125 < Z < 0.75)$

$= \phi(0.75) - (1 - \phi(1.125))$

$= 0.773 - (1 - \underline{0.86971})$ مش من الجدول (القيم موجودة في الجدول)

$= 0.642$

Q12: if $M(t_1, t_2)$ is the m.g.f of a bivariate normal distribution, compute the coveriance by using the formula $\frac{\partial^2 M(0,0)}{\partial t_1 \partial t_2} - \frac{\partial M(0,0)}{\partial t_1} \frac{\partial M(0,0)}{\partial t_2}$

Now let $\psi(t_1, t_2) = \ln M(t_1, t_2)$.

show that $\frac{\partial^2 \psi(0,0)}{\partial t_1 \partial t_2}$ gives this covariance directly.

$$\text{covariance} = \rho \sigma_1 \sigma_2 - \cancel{\mu_1 \mu_2} + \cancel{\mu_1 \mu_2}$$
$$= \rho \sigma_1 \sigma_2$$

$\rightarrow$ let $\psi(t_1, t_2) = \ln M(t_1, t_2)$

$\rightarrow$ Now : $$\left.\frac{\partial \psi(t_1, t_2)}{\partial t_1}\right|_{(0,0)} = \frac{1}{M(t_1, t_2)} \cdot \left.\frac{\partial M(t_1, t_2)}{\partial t_1}\right|_{(0,0)}$$

$$= \frac{1}{M(0,0)} \cdot \frac{\partial M(0,0)}{\partial t_1}$$

$\rightarrow$ Hence, $$\frac{\partial \psi(t_1, t_2)}{\partial t_1} = \mu_1$$

$\rightarrow$ Now, $$\left.\frac{\partial^2 \psi(t_1, t_2)}{\partial t_1 \partial t_2}\right|_{(0,0)} = \left.\frac{\partial^2 M(0,0)}{\partial t_1 \partial t_2} - \frac{\partial M(0,0)}{\partial t_1} \cdot \frac{\partial M(0,0)}{\partial t_2}\right|_{(0,0)}$$

$$= \rho \sigma_1 \sigma_2$$

$$= \text{covariance}.$$

Q73: let X and y have a bivariate normal distribution with parameters $\mu_1 = 5$, $\mu_2 = 10$, $\sigma_1^2 = 1$, $\sigma_2^2 = 25$ and $\rho > 0$, if $Pr(4 < y < 16 \mid x = 5)$ equal 0.954, determine ρ.

$$y|x \sim N\left(\mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x - \mu_1), \ \sigma_2^2(1-\rho^2)\right)$$

$$y|x \sim N\left(10 + \rho \frac{5}{1}\cancel{(5-5)}^{0}, \ 25(1-\rho^2)\right)$$

$$y|x \sim N\left(10, \ 25(1-\rho^2)\right).$$

so $$Z = \frac{y - 10}{\sqrt{25(1-\rho^2)}} = \frac{y-10}{5\sqrt{1-\rho^2}}$$

$$\Rightarrow Pr(4 < y < 16 \mid x = 5) = Pr\left(\frac{-6}{5\sqrt{1-\rho^2}} < Z < \frac{6}{5\sqrt{1-\rho^2}}\right) = 0.954.$$

$$0.954 = \phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) - \phi\left(\frac{-6}{5\sqrt{1-\rho^2}}\right)$$

$$0.954 = \phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) - 1 + \phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right)$$

$$0.954 = 2\phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right) - 1$$

$+1$ $+1$

$\div 2$

$$0.977 = \phi\left(\frac{6}{5\sqrt{1-\rho^2}}\right)$$

$$\phi^{-1}(0.977) = \frac{6}{5\sqrt{1-\rho^2}} \rightarrow \frac{2}{1} = \frac{6}{5\sqrt{1-\rho^2}}$$

$$10\sqrt{1-\rho^2} = 6$$

$$\left(\sqrt{1-\rho^2} = 0.6\right)^2$$

$$1-\rho^2 = 0.36$$

$$\rho^2 = 0.64 \Rightarrow \boxed{\rho = 0.8} = \frac{4}{5}$$

ما بناخذ السالب لأنه $\rho > 0$ zero

Q74: let X and y have a bivariate Normal distribution with parameters $\mu_1 = 20$, $\mu_2 = 40$, $\sigma_1^2 = 9$, $\sigma_2^2 = 4$ and $\rho = 0.6$. Find the shortest interval for which 0.9 is the conditional probability that y is in this interval given that $X = 22$. $y|X=22$

→ Find the mean of conditional prob.

$$\mu_2 + \rho \frac{\sigma_2}{\sigma_1}(x - \mu_1) = 40 + 0.6\left(\frac{2}{3}\right)(22 - 20)$$

$$= 40.8$$

→ Find the variance of conditional prob.

$$\sigma_2^2 (1 - \rho^2) = 4(1 - 0.6^2)$$

$$= 2.56$$

& the standard deviation = 1.6.

→ The interval :

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