

Rayyan Masri

1191184

H.W. 1

Ch. 2: Groups

Q.4] $GL(2, \mathbb{R})$, non-abelian :-

let $A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$, $|A| = 2 \neq 0$

and $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $|B| = 1 \neq 0$

$AB \neq BA$

$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$, non-abelian

Q.6] $a^{-1}ba \neq b$?

$G(\mathbb{Z}_2, \oplus)$

	0	1
0	0	1
1	1	0

let $a = 0 \rightarrow a^{-1} = 0$

and let $b = 1$

$a^{-1}ba = (0)(1)(0) = 0 \neq 1 \neq b$.

Q.11] ① closed, $\det(AB) = (\det A)(\det B) = 1 \cdot 1 = 1 \Rightarrow AB \in G$ ✓

② Identity, yes, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓ since $|I| = 1$

③ Inverse :- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ its determinant is $ad - bc = 1$ ✓

④ Matrix multiplication is associative

$\rightarrow$ is a group matrix.

$A, B, C \in G$

$(A \cdot B) \cdot C = A \cdot (B \cdot C)$ ✓

Q.12] any integer $n \geq 2$:-

$$U(n) = \{1, \dots, n-1\}$$

$$1^2 = 1, \quad (n-1)^2 = 1 \quad \rightarrow \quad (n-1)^2 = n^2 - 2n + 1$$

ex. $U(9) = \{1, \dots, 8\}$

$$1^2 = 1, \quad 8^2 = 1$$

$$= (n-2)n + 1$$

$$= 1, \text{ since } n \equiv 0 \text{ always.}$$

Q.14] G is group with the following property whenever a, b and c belong to G and $ab = ca$ then $b = c$, prove G is abelian:-

$$\forall a, b \in G$$

$$aba = aba \quad (\text{cancel from left and right}).$$

$$ba = ab$$

Q.17] G is abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$:-

$\Rightarrow$ suppose that G is abelian

$$\Rightarrow (ab)^{-1} = b^{-1}a^{-1} = a^{-1}b^{-1}$$

$\Leftarrow$ if $(ab)^{-1} = a^{-1}b^{-1}$

$$\Rightarrow (ab)(a^{-1}b^{-1}) = e \quad (\text{multiplying both sides on the right by } ba)$$

$$\Rightarrow ba = ab$$

$\leadsto$ abelian

$$(ab)(a^{-1}b^{-1})(ba) = ba$$

$$\underline{ab} \underline{ba} = \underline{ba}$$

Q.18] $(a^{-1})^{-1} = a$ for all a :-

$$aa^{-1} = e \quad (\text{multiplying from right by } (a^{-1})^{-1})$$

$$aa^{-1}(a^{-1})^{-1} = e(a^{-1})^{-1}$$

$$a(a^{-1}(a^{-1})^{-1}) = (a^{-1})^{-1}$$

$$ae = (a^{-1})^{-1}$$

$$\Rightarrow a = (a^{-1})^{-1} \quad \#$$

Q.20] If $a_1, a_2, \dots, a_n$ belong to a group
what is the inverse of $a_1 a_2 \dots a_n$?

$$\Rightarrow (a_1 a_2 \dots a_{n-1} a_n)^{-1} = a_n^{-1} a_{n-1}^{-1} \dots a_2^{-1} a_1^{-1}$$

$$\Rightarrow (a_1 a_2 \dots a_{n-1} a_n) (a_n^{-1} a_{n-1}^{-1} \dots a_2^{-1} a_1^{-1}) = e$$

Q.24] $U(12) = \{1, 5, 7, 11\}$

	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

Q.25]

	e	a	b	c	d
e	e	a	b	c	d
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c

Q.26] if $(ab)^2 = a^2 b^2$ in a group G then $ab = ba$:-

$$\Rightarrow (ab)^2 = a^2 b^2$$

$$abab = aabb$$

$$\Rightarrow ba = ab$$

Q.28] ① closure: $(3^m 6^n)(3^s 6^t) = 3^{m+s} 6^{n+t} \in G$

② associativity ✓

$$(3^{n_1} 6^{n_1} \cdot 3^{n_2} 6^{n_2}) \cdot 3^{n_3} 6^{n_3} = 3^{n_1+n_2} 6^{n_1+n_2} \cdot 3^{n_3} 6^{n_3} = 3^{n_1+n_2+n_3} 6^{n_1+n_2+n_3}$$

③ identity: $3^0 6^0 = 1 = e$

④ Inverse: $(3^m 6^n)^{-1} = 3^{-m} 6^{-n}$, $3^m 6^n \cdot 3^{-m} 6^{-n} = 1 = e$

$\Rightarrow$ group.

Q.35] G is group with property that the square of every element is the identity, then G is abelian :-

$$\Rightarrow \text{since } a^2 = b^2 = (ab)^2 = e, \quad a, b \in G \quad \leftarrow \text{closure.}$$

$$\Rightarrow aab b = abab$$

$$\Rightarrow ab = ba$$

$\Rightarrow$ abelian

Q.39] $G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} \mid a \in \mathbb{R}, a \neq 0 \right\}$ &

$$\Rightarrow \begin{bmatrix} a & a \\ a & a \end{bmatrix} \begin{bmatrix} b & b \\ b & b \end{bmatrix} = \begin{bmatrix} 2ab & 2ab \\ 2ab & 2ab \end{bmatrix}$$

and $2ab \neq 0 \rightarrow$ closure: (1)

$\Rightarrow$ matrix multiplication is associative. (2)

$$\Rightarrow \text{the identity is } \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \quad (3)$$

$$\Rightarrow \text{the inverse } \begin{bmatrix} a & a \\ a & a \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{4a} & \frac{1}{4a} \\ \frac{1}{4a} & \frac{1}{4a} \end{bmatrix} \quad (4)$$

$\Rightarrow$ the group $GL(2, \mathbb{R})$ has different identity than the group G .