

10.5 The ratio & Root tests.

7 Use the ratio test to determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n^2 (n+2)!}{n! 3^{2n}}$$

$$\frac{n^2 (n+2)!}{n! 3^{2n}} > 0 \text{ for all } n \geq 1.$$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \left[\frac{(n+1)^2 ((n+1)+2)!}{(n+1)! 3^{2(n+1)}} \cdot \frac{n^2 (n+2)!}{n! 3^{2n}} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^2 (n+3)!}{(n+1)! 3^{2n+2}} \cdot \frac{n! 3^{2n}}{n^2 (n+2)!}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^2 (n+3) \cancel{(n+2)!}}{(n+1) \cdot \cancel{n!} \cdot 3^2 \cdot n^2 \cancel{(n+2)!}} \cdot \cancel{n!} \cdot 3^{2n} \cdot \cancel{3^{2n}}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^2 (n+3)}{9(n+1) n^2}$$

$$= \boxed{\frac{1}{9}} < 1$$

by ratio test, the series is convergent.

(12) $\sum_{n=1}^{\infty} \left(\ln \left(e^2 + \frac{1}{n} \right) \right)^{n+1}$ use root test.

$$\left[\ln \left(e^2 + \frac{1}{n} \right) \right]^{n+1} \geq 0 \quad \text{for all } n \geq 1$$

$$\text{now, } \lim_{n \rightarrow \infty} \sqrt[n]{\left[\ln \left(e^2 + \frac{1}{n} \right) \right]^{n+1}}$$

$$= \lim_{n \rightarrow \infty} \left[\ln \left(e^2 + \frac{1}{n} \right) \right]^{\frac{n+1}{n}}$$

$$= \lim_{n \rightarrow \infty} \left[\ln \left(e^2 + \frac{1}{n} \right) \right]^{1 + \frac{1}{n}}$$

$$= \left[\ln (e^2 + 0) \right]^{1+0}$$

$$= 2 > 1$$

by the root test the series diverges.

(16) $\sum_{n=2}^{\infty} \frac{1}{n^{1+n}}$, use root test.

$$\frac{1}{n^{1+n}} \geq 0 \quad \text{for all } n \geq 2.$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{1}{n^{1+n}}} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{1}}{\sqrt[n]{n^{1+n}}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n^{\frac{1}{n}} \cdot n^{\frac{n}{n}}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{n} \cdot n} = 0 < 1$$

So, by the root test, the series converges.

20 Use any method to determine if the series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n!}{10^n}$$

Let use the ratio test.

$$\begin{aligned}\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} &= \lim_{n \rightarrow \infty} \frac{(n+1)!}{10^{n+1}} \\&= \lim_{n \rightarrow \infty} \frac{(n+1) \cdot \cancel{n!}}{\cancel{10^n} \cdot 10} \cdot \frac{\cancel{10^n}}{\cancel{n!}} \\&= \lim_{n \rightarrow \infty} \frac{(n+1)}{10} = \infty > 1\end{aligned}$$

The series diverges by the ratio test.

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$$\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$$

by root test, $\lim_{n \rightarrow \infty} \sqrt[n]{\frac{(\ln n)^n}{n^n}} = \lim_{n \rightarrow \infty} \frac{\ln n}{n} = \frac{\infty}{\infty}$

$$= \lim_{n \rightarrow \infty} \frac{1}{1} = 0 < 1$$

So, The series converges.

38 $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Let use ratio test $\therefore$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \left[\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1) \cancel{n!}}{(n+1)^n (n+1) \cancel{n!}} \cdot \frac{n^n}{\cancel{n!}} = \lim_{n \rightarrow \infty} \frac{n^n}{(n+1)^n}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{\frac{(n+1)^n}{n^n}} = \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{n+1}{n}\right)^n}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e}$$

$$= \frac{1}{e} < 1$$

The series converges.

43 $\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

using ratio test $\therefore$

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{((n+1)!)^2}{(2(n+1))!} \cdot \frac{(2n)!}{(n!)^2}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^2 \cancel{(n!)^2}}{(2n+2)(2n+1) \cancel{(2n)!}} \cdot \frac{(2n)!}{\cancel{(n!)^2}}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{1}{4} < 1$$

So, the series converges.

46 $a_1 = 1$, $a_{n+1} = \frac{1 + \tan^{-1} n}{n} a_n$.

Let use ratio test.

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(1 + \tan^{-1} n)(a_n)}{n a_n}$$

$$= \lim_{n \rightarrow \infty} \frac{(1 + \tan^{-1} n) \cancel{a_n}}{n} \cdot \frac{1}{\cancel{a_n}}$$

$$= \frac{1 + \frac{\pi}{2}}{\lim_{n \rightarrow \infty} n} = 0 < 1$$

$\therefore$ The series converges.

60 $\sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2}$

Using root test.

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{n^n}{(2^n)^2}} = \lim_{n \rightarrow \infty} \frac{n^{\frac{n}{n}}}{2^{\frac{2n}{n}}}$$

$$= \lim_{n \rightarrow \infty} \frac{n}{2^2}$$

$$= \lim_{n \rightarrow \infty} \frac{n}{4}$$

$$= \infty$$

The series diverges.

10.6 Alternating Series, Absolute & Conditional Converges.

8 Determine if the following series converges or diverges.

$$\sum_{n=1}^{\infty} (-1)^n \cdot \frac{10^n}{(n+1)!}$$

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{10^n}{(n+1)!}$$

Using Ratio Test:-

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{10^{n+1}}{(n+1+1)!} \cdot \frac{(n+1)!}{10^n}$$

$$= \lim_{n \rightarrow \infty} \frac{10 \cdot 10}{(n+2)(n+1)!} \cdot \frac{(n+1)!}{10}$$

$$= \lim_{n \rightarrow \infty} \frac{10}{n+2} = 0 < 1$$

So, $\sum_{n=1}^{\infty} |a_n|$ converges

The series converges absolutely $\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{10^n}{(n+1)!}$ converges
by the Absolute convergence test.

20
$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n!}{2^n}$$

by nth term test

$$\lim_{n \rightarrow \infty} \frac{n!}{2^n} = \frac{1}{\lim_{n \rightarrow \infty} \left(\frac{2^n}{n!} \right)} = \frac{1}{0} = \infty$$

The series diverges.

$$\boxed{13} \quad \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n} + 1}{n+1}$$

Using the alternating series test:

$$1) \quad U_n = \frac{\sqrt{n} + 1}{n+1} > 0 \quad \forall n \geq 1.$$

$$2) \quad U_n \geq U_{n+1}$$

$$3) \quad \lim_{n \rightarrow \infty} U_n = \lim_{n \rightarrow \infty} \frac{\sqrt{n} + 1}{n+1} = \boxed{0}$$

So, the series converges by
alternating series test.

$$\text{Let } f(x) = \frac{\sqrt{x} + 1}{x+1}, \quad x \geq 1.$$

$$f'(x) = \frac{x+1}{2\sqrt{x}} - \frac{(\sqrt{x} + 1)}{(x+1)^2}$$

$$= \frac{x+1 - 2x - 2\sqrt{x}}{2\sqrt{x}(x+1)^2}$$

$$= \frac{-x - 2\sqrt{x} + 1}{2\sqrt{x}(x+1)^2} < 0, \quad \forall x \geq 1$$

$f(x)$ is decreasing

25 Determine if the series converges absolutely, conditionally or diverges.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1+n}{n^2}$$

★ Converges Absolutely ??

$$\begin{aligned} \sum_{n=1}^{\infty} |a_n| &= \sum_{n=1}^{\infty} \frac{1+n}{n^2} \\ &= \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^2}}_{\substack{\text{converges} \\ \text{(p-series)} \\ \text{p} \geq 1}} + \underbrace{\sum_{n=1}^{\infty} \frac{1}{n}}_{\substack{\text{diverges} \\ \text{(harmonic series)}}} = \text{diverges.} \end{aligned}$$

⇒ The series does not converge absolutely.

★ Converges Conditionally ??

1] let $u_n = \frac{1+n}{n^2} > 0$

2] ⇒ $u_n > u_{n+1} > 0 \quad \forall n \geq 1$

3] $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1+n}{n^2} = 0$

$$f(x) = \frac{1+x}{x^2} = \frac{1}{x^2} + \frac{1}{x}, \quad x \geq 1$$

$$f'(x) = -\left(\frac{2}{x^3} + \frac{1}{x^2}\right) < 0 \quad \forall x \geq 1$$

$f(x)$ is decreasing.

⇒ Converges by alternating series test.

⇒ So, The series converges conditionally but not converges absolutely.

30 $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n - \ln n}$

★ Converges absolutely?? (No)

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{\ln n}{n - \ln n}$$

let use comparison test & compare with $\frac{1}{n}$.

$$n - \ln n < n.$$

$$\frac{1}{n - \ln n} > \frac{1}{n} \Rightarrow \frac{\ln n}{n - \ln n} > \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ (diverges - harmonic series).}$$

So, by direct comparison test $\sum_{n=1}^{\infty} \frac{\ln n}{n - \ln n}$ diverges.

★ Converges conditionally?? (yes).

1] $U_n = \frac{\ln n}{n - \ln n} \geq 0 \quad \forall n \geq 1.$

2] $f(x) = \frac{\ln x}{x - \ln x}, \quad f'(x) = \frac{x - \ln x - \ln x (1 - \frac{1}{x})}{(x - \ln x)^2}$

$$= \frac{x - \cancel{\ln x} - x \ln x - \cancel{\ln x}}{x (x - \ln x)^2}$$

$$= \frac{x (1 - \ln x)}{x (x - \ln x)^2} < 0 \quad \forall x \geq e$$

So, $U_n \geq U_{n+1} \quad \forall n \geq 3.$

3] Now, $\lim_{n \rightarrow \infty} U_n = \lim_{n \rightarrow \infty} \frac{\ln n}{n - \ln n} = \lim_{n \rightarrow \infty} \frac{\frac{1}{n}}{1 - \frac{1}{n}} = 0$

$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n - \ln n}$ (converges by alternating series test).

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$$\sum_{n=1}^{\infty} \frac{(-1)^n (2n)!}{2^n n! n}$$

diverges by n-th term test

since $\lim_{n \rightarrow \infty} |a_n| = \lim_{n \rightarrow \infty} \frac{(2n)!}{2^n n! n}$

$$= \lim_{n \rightarrow \infty} \frac{\cancel{(2n)} (2n-1) (2n-2) \dots (n+1) \cancel{n!}}{2^n \cancel{n!} \cancel{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{(2n-1) (2n-2) \dots (n+1)}{2^{n-1}}$$

$$> \lim_{n \rightarrow \infty} \left(\frac{n+1}{2} \right)^{n-1}$$

$$= \infty \neq 0.$$

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$$\sum_{n=1}^{\infty} (-1)^n (\sqrt{n^2+n} - n)$$

Let use n-th term test

$$\lim_{n \rightarrow \infty} (\sqrt{n^2+n} - n) = \lim_{n \rightarrow \infty} (\sqrt{n^2+n} - n) \cdot \frac{\sqrt{n^2+n} + n}{\sqrt{n^2+n} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{\cancel{n^2} + n - \cancel{n^2}}{\sqrt{n^2+n} + n} = \lim_{n \rightarrow \infty} \frac{n}{n\sqrt{1+\frac{1}{n}} + n}$$

$$= \lim_{n \rightarrow \infty} \frac{\cancel{n}}{\cancel{n}(\sqrt{1+\frac{1}{n}} + 1)} = \lim_{n \rightarrow \infty} \frac{1}{\sqrt{1+\frac{1}{n}} + 1} = \frac{1}{1+1} = \frac{1}{2} \neq 0$$

The series diverges.

50 Estimate the magnitude of the error involved in using the sum of the first four terms to approximate the sum of the series.

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{10^n}$$

$$|\text{error}| < u_{4+1} = |a_{4+1}|$$

$$|\text{error}| < |a_5| = \left| (-1)^{5+1} \frac{1}{10^5} \right|$$

$$|\text{error}| < 0.00001$$

54 Determine how many terms should be used to estimate the sum of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$ with an error of less

than 0.001

$$|\text{error}| < 0.001$$

$$\Rightarrow u_{n+1} < 0.001$$

$$\frac{n+1}{(n+1)^2+1} < 0.001$$

$$(n+1) 1000 < (n+1)^2 + 1$$

$$1000n + 1000 < n^2 + 2n + 2$$

$$0 < n^2 - 998n - 998$$

$$n > \frac{-(-998) \pm \sqrt{(-998)^2 - 4(1)(-998)}}{2(1)} \approx 998.99899$$

$$n \geq 999$$