# BIBO Stability (Boundary Input Boundary output) Stability

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## Input Output Stability of LTI Systems

> **Theorem (BIBO Stability).** A LTI system with proper rational transfer matrix $\mathbf{G}(s) = [\mathbf{G}_{ij}(s)]$ is BIBO stable if and only if every pole of every entry $\mathbf{G}_{ij}(s)$ of $\mathbf{G}(s)$ has negative real part.

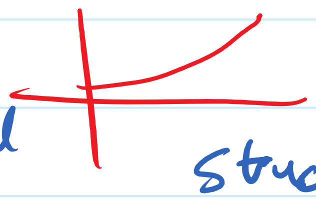
* This type of stability study the stability of the system under the effect of input ($u(t)$)

BIBO stability < stable / unstable

~~t~~ $u(t) \longrightarrow \boxed{\text{Sys}} \longrightarrow x(t)$ ?

Ex:-

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$



study the BIBO stability

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$$G(s) = C(sI - A)^{-1} B + D$$

$$(sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} s & -1 \\ -1 & s-2 \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} s-2 & 1 \\ 1 & s \end{bmatrix} \frac{1}{s^2 - 2s - 1}$$

$$C(sI - A)^{-1} = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{1\times 2} \underbrace{\begin{bmatrix} \frac{s-2}{\Delta} & \frac{1}{\Delta} \\ \frac{1}{\Delta} & \frac{s}{\Delta} \end{bmatrix}}_{2\times 2} = \begin{bmatrix} \frac{s-2}{\Delta} & \frac{1}{\Delta} \end{bmatrix}$$

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$$C(SI-A)^{-1} = \begin{bmatrix} \frac{s-2}{\Delta} & \frac{1}{\Delta} \end{bmatrix}$$

$$C(SI-A)^{-1}B = \underbrace{\begin{bmatrix} \frac{s-2}{\Delta} & \frac{1}{\Delta} \end{bmatrix}}_{1\times 2} \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{2\times 1}$$

$$= \frac{1}{\Delta} = \frac{1}{s^2-2s-1}$$

$s^2-2s-1=0$ → $s_1 = -0.4142$, $s_2 = 2.4142$

unstable in the sense of BIBO stability

$$G(s) = \begin{bmatrix} \frac{1}{(s-2)(s+2)} & \frac{1}{(s+1)} \\ \frac{1}{(s+10)} & \frac{1}{(s+3)} \end{bmatrix}$$

Transfer matrix

$g_{11}$: $s_1 = 2$, $s_2 = -2$

$g_{12} = -1$

$g_{21} = -1$

$g_{22} = -3$

Based on that the sys is unstable in the sense of BIBO stability.

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**BIBO stability of state equations.** When the system is represented by state equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$$

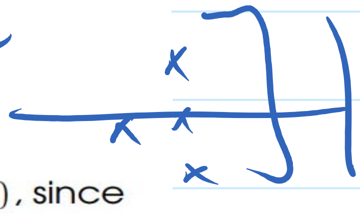
$\lambda_1 = -2$

$\lambda_2 = -3$

$\lambda_{3,4} = -2 \pm 3j$

the BIBO stability will depend on the eigenvalues of the matrix $\mathbf{A}$, since every pole of $\mathbf{G}(s)$ is an eigenvalue of $\mathbf{A}$.

stable



## Input Output Stability of LTI Systems

Note that **not every eigenvalue of $\mathbf{A}$ is a pole of** $\mathbf{G}(s)$, since there may be pole-zero cancellations while computing $\mathbf{G}(s)$. Thus, a state equation may be BIBO stable even when some eigenvalues of $\mathbf{A}$ do not have negative real part.

## Input Output Stability of LTI Systems

Note that **not every eigenvalue of $\mathbf{A}$ is a pole of** $\mathbf{G}(s)$, since there may be pole-zero cancellations while computing $\mathbf{G}(s)$. Thus, a state equation may be BIBO stable even when some eigenvalues of $\mathbf{A}$ do not have negative real part.

**Example.** Although the system

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} -2 \\ 0 \end{bmatrix} \mathbf{u}(t)$$

$$\mathbf{y}(t) = \begin{bmatrix} -2 & 3 \end{bmatrix} \mathbf{x}(t) - 2\mathbf{u}(t)$$

has one eigenvalue with positive real part $\lambda = 1$, it is BIBO stable, since its transfer function

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \frac{2(1-s)}{(s+1)}$$

has a single pole at $s = -1$. □

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# Internal Stability "Lyapunov stability"

$$\left.\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}\right\} \text{LTI}$$

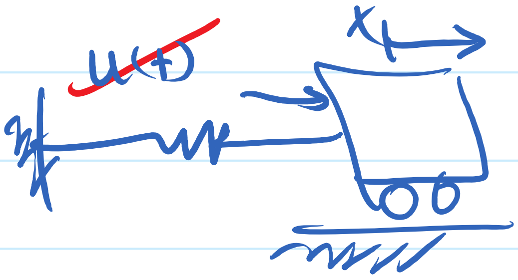
$x(0)$ ↓ $\boxed{\text{Sys}}$ IC

study the stability under the effect of IC $x(0)$ . when $u(t)=0$

$$\left.\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned}\right\} u(t)=0$$

$|\lambda I - A| = 0$

Internal stability → unstable / m.stable / Asy.stable




$Z = \begin{bmatrix} 0 & 1 \\ 0 & \end{bmatrix}$

$Z_1 = x$

$Z_2 = \dot{x}$

Ex: Study the BIBO stability and the lyponav stability for the sys. shown behind?

$$\dot{x}(t) = \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} -2 & 3 \end{bmatrix} x(t) - 2u(t)$$

* internal stability

$$|\lambda I - A| = \left| \begin{bmatrix} \lambda+1 & -10 \\ 0 & \lambda-1 \end{bmatrix} \right| = (\lambda+1)(\lambda-1) = 0$$

$\lambda_1 = 1$

$\lambda_2 = -1$


unstable

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x in the sense of BIBO Stability

$$\dot{x}(t) = \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} -2 & 3 \end{bmatrix} x(t) - 2u(t)$$

$$[sI - A] = \begin{bmatrix} s+1 & -10 \\ 0 & s-1 \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} s-1 & 10 \\ 0 & s+1 \end{bmatrix} \frac{1}{s^2-1}$$

$$C(sI - A)^{-1} = \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} \frac{s-1}{\Delta} & \frac{10}{\Delta} \\ \frac{0}{\Delta} & \frac{s+1}{\Delta} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-2(s-1)}{\Delta} & \frac{(3s-17)}{\Delta} \end{bmatrix}$$

$$C(sI - A)^{-1} B = \begin{bmatrix} \frac{-2(s-1)}{\Delta} & \frac{(3s-17)}{\Delta} \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$= \frac{+4(s-1)}{s^2-1} = \frac{4\cancel{(s-1)}}{\cancel{(s-1)}(s+1)}$$

$s = -1$ (stable) in the sense of BIBO