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MATH1321

Calculus 2

Chapter 10.3

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10.3 The Integral Test (conv./div.) IT

consider $\sum_{n=k}^{\infty} a_n$, where:

1) a_n positive terms. 2) $a_n = f(n)$ is cont., positive and decreasing on $[k,\infty)$.

Then $\sum_{n=k}^{\infty} a_n$ and $\int_k^{\infty} f(x)\,dx$ are both conv. or both div.

Ex. Does the following series conv. or div. ?

Reminder

$$\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & \text{if } p>1 \\ \infty & \text{if } p\leq 1 \end{cases} = \begin{cases} \text{conv. if } p>1 \\ \text{Div. if } p\leq 1 \end{cases}$$

$$\int_0^{1} \frac{dx}{x^p} = \begin{cases} \frac{1}{1-p} & \text{if } p<1 \\ \infty & \text{if } p\geq 1 \end{cases} = \begin{cases} \text{conv. if } p<1 \\ \text{Div. if } p\geq 1 \end{cases}$$

1) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $a_n = \frac{1}{n^2}$ positive terms $\forall n = 1,2,3,\dots$

$f(x) = \frac{1}{x^2}$ is positive, cont. and decreasing on $[1,\infty)$.

$\int_1^{\infty} \frac{1}{x^2}\,dx = \frac{1}{2-1} = 1$ the improper integral conv. to 1.

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ conv. by Integral test.

(we can't say conv. to 1)

2) $\sum_{n=1}^{\infty} \frac{1}{n^3}$ conv. by IT.

Test: P- Series Test

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ conv. if $p>1$

div. if $p\leq 1$

3) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ div. by P-series test.

4) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ $a_n = \frac{1}{n^2+1}$ positive term $\forall n = 1,2,\dots$

$f(x) = \frac{1}{x^2+1}$ is positive, cont. and decreasing on $[1,\infty)$.

$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2+1}\,dx = \lim_{b\to\infty} \tan^{-1}x \Big|_1^b$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$. the improper integral conv.

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ conv. by Integral test.

5) $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ $a_n = \frac{1}{2n-1}$ positive $\forall n = 1,2,\dots$

$f(x) = \frac{1}{2x-1}$ is positive, cont. and decreasing on $[1,\infty)$.

$\int_1^{\infty} \frac{dx}{2x-1} = \lim_{b\to\infty} \int_1^b \frac{1}{2x-1}\,dx = \frac{1}{2} \lim_{b\to\infty} \ln|2x-1| \Big|_1^b$

$= \frac{1}{2} \ln \lim_{b\to\infty} 2b-1 = \frac{1}{2} \cdot \ln\infty = \infty$.

The improper integral div. Hence, $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ div. by integral test.

7) $\sum_{n=3}^{\infty} \frac{5-n}{3n-\frac{1}{2}}$ $\lim_{n\to\infty} \frac{5-n}{3n-\frac{1}{2}} = -\frac{1}{3}$

div. by n^{th} term test.

6) $\sum_{n=1}^{\infty} n\sin\frac{1}{n}$. $\lim_{n\to\infty} n\sin\frac{1}{n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}}$ let $u = \frac{1}{n}$

$= \lim_{u\to 0^+} \frac{\sin u}{u} = 1$.

div. by n^{th} term test.

8) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}(\sqrt{n}+1)}$

$a_n = \frac{1}{\sqrt{n}(\sqrt{n}+1)}$ positive terms $\forall n = 1, 2, \ldots$

$f(x) = \frac{1}{\sqrt{x}(\sqrt{x}+1)}$ is positive, cont. and decreasing on $[1, \infty)$.

$\int_1^{\infty} \frac{dx}{\sqrt{x}(\sqrt{x}+1)} = \lim_{b\to\infty} \int_1^b \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$ let $u = \sqrt{x}+1$

$= \lim_{b\to\infty} \int \frac{2}{u} du = 2 \lim_{b\to\infty} \ln|\sqrt{x}+1| \Big|_1^b$ $2\sqrt{x}\,du = dx$

$= 2 \ln \lim_{b\to\infty} \frac{b+1}{2} = 2 \cdot \infty = \infty$.

Hence, $\sum_{n=1}^{\infty} a_n$ div. by integral test.

9) $\sum_{n=1}^{\infty} n \tan\frac{1}{n}$ $\lim_{n\to\infty} \frac{\tan\frac{1}{n}}{\frac{1}{n}} = 1$

div. by n^{th} term test.

10) $\sum_{n=2}^{\infty} \frac{\ln n^2}{n}$

$a_n = \frac{\ln n^2}{n}$ positive term $\forall n = 2, 3, \ldots$

$f(x) = \frac{\ln x^2}{x}$ is positive, cont. and $\downarrow$ on

$= \frac{\ln 2^2}{2} + \sum_{n=3}^{\infty} \frac{\ln n^2}{n}$ $f'(x) = \frac{x(2\frac{1}{x}) - \ln x^2}{x^2} = \frac{2 - \ln x^2}{x^2}$

b_n +, cont., $\downarrow$ on $[3, \infty)$ $2 - \ln x^2 = 0 \rightarrow 2 = 2\ln x$

$\int_3^{\infty} \frac{\ln x^2}{x} dx = \lim_{b\to\infty} \int_3^b 2 \frac{\ln x}{x} dx$ $\ln x = 1 \rightarrow x = e$.

let $u = \ln x \rightarrow x\,du = dx$

$+ \; + \; + \; | \; - \; - \; -$

e

$= \lim_{b\to\infty} 2 \int u\,du = \cancel{2} \lim_{b\to\infty} \frac{u^2}{\cancel{2}} \Big|_3^b$ ≈ 2.718 so $f(x) \downarrow$ on $[e, \infty)$.

$= \lim_{b\to\infty} (\ln b)^2 - (\ln 3)^2 = \infty$.

Hence, $\frac{\ln 2^2}{2} + \infty = \ln 2 + \infty = \infty$ so $\sum_{n=2}^{\infty} a_n$ div.

by integral test.