Electromagnetic Theory I

Abdallah Sayyed-Ahmad

Department of Physics

Birzeit University

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Chapter 1: Vector Analysis

- Vector Algebra
- Differential Calculus
- Integral Calculus
- Curvilinear Coordinates
- The Dirac Delta Function
- The Theory of Vector Fields

Differential Calculus

(i) Gradient

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$$\left(\frac{\partial f}{\partial x}\hat{x} + \frac{\partial f}{\partial y}\hat{y} + \frac{\partial f}{\partial z}\hat{z}\right) \cdot (dx\hat{x} + dy\,\hat{y} + dz\,\hat{z}) = \vec{\nabla} f \cdot d\vec{l}$$

Gradient of f

What is the physical meaning of the gradient?

$$\vec{\nabla} f \cdot d\vec{l} = |\vec{\nabla} f||d\vec{l}|\cos\theta$$

For a fixed distance $|d\vec{l}|$, the greatest change in f occurs along the direction of the gradient

**Gradient is a vector that points in the direction of maximum increase of a function. Its magnitude gives the slope (rate of increase) along this maximal direction.

Gradient

Example: find the gradient of the magnitude of the position vector

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{r} \qquad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}} = \frac{y}{r} \qquad \frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2 + y^2 + z^2}} = \frac{z}{r}$$

$$\rightarrow \vec{\nabla} r = \frac{x}{r}\hat{x} + \frac{y}{r}\hat{y} + \frac{z}{r}\hat{z} = \frac{1}{r}(x\,\hat{x} + y\,\hat{y} + z\,\hat{z}) = \frac{\vec{r}}{r} = \hat{r}$$

The distance from the origin increases most rapidly in the radial direction

The del operator ($\vec{\nabla}$)

$$\vec{\nabla}=\left(\frac{\partial}{\partial x}\hat{x}+\frac{\partial}{\partial y}\hat{y}+\frac{\partial}{\partial z}\hat{z}\right)$$

A vector operator (not a vector)

It can act on other scalar/vector functions as follows

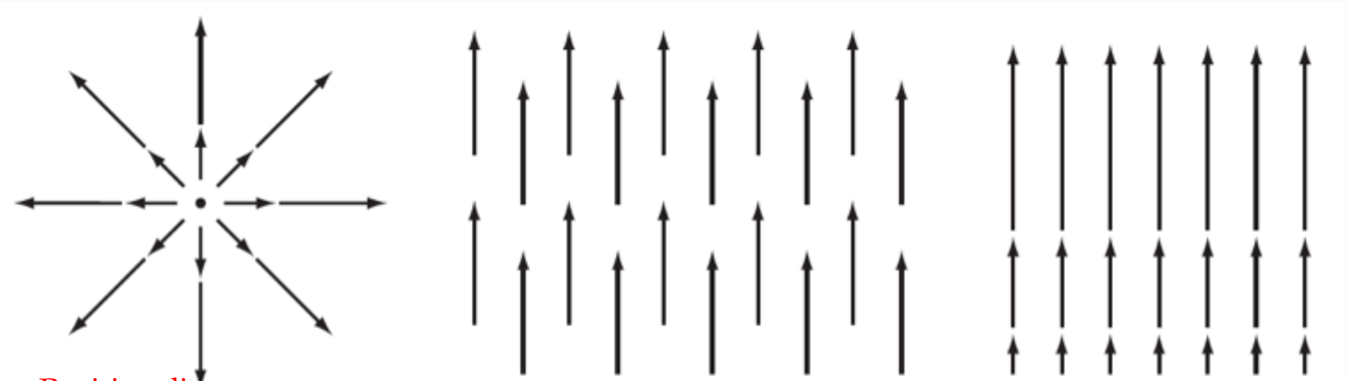
- On scalar function f (gradient): $\vec{\nabla} f$
- On a vector function via dot product (divergence): $\vec{\nabla}\cdot\vec{v}$
- On a vector function via cross product (curl): $\vec{\nabla}\times\vec{v}$

The divergence ($div\,(\vec{v})$ or $\vec{\nabla}\cdot\vec{v}$)

For the vector field, $\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$

$$\vec{\nabla}\cdot\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

A scalar, a measure of how much the vector field spread out (diverges) from the point of interest



Positive divergence zero divergence Positive divergence

The divergence

Example: find the divergence of the following three vector fields: $(a)\ \vec{v} = \vec{r}$ $(b)\ \vec{v} = \hat{z}$ $(c)\vec{v} = z\,\hat{z}$ $(d)\vec{v} = y\,\hat{x} - x\hat{y}$

$$(a)\ \vec{\nabla} \cdot \vec{v} = \vec{\nabla} \cdot \vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

$$(b)\ \vec{\nabla} \cdot \vec{v} = 0$$

$$(c)\vec{\nabla} \cdot \vec{v} = \frac{\partial z}{\partial z} = 1$$

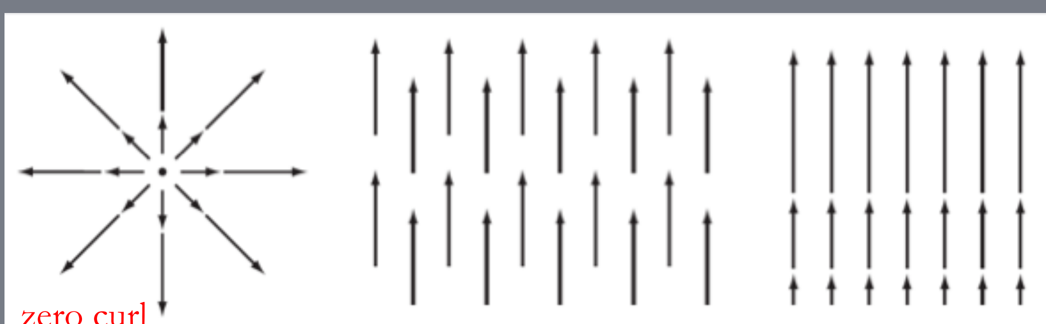
$$(d)\vec{\nabla} \cdot \vec{v} = \frac{\partial y}{\partial x} - \frac{\partial x}{\partial y} = 0$$

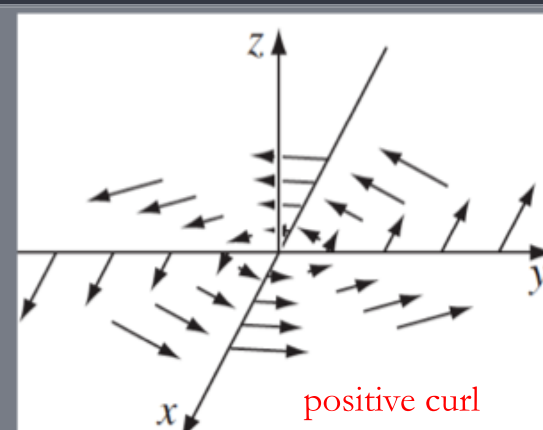
The curl ($rot(\vec{v})$ or $\vec{\nabla}\times\vec{v}$)

For the vector field, $\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

A scalar, a measure of how much the vector field curl(or rotates) around the point of interest





The curl

Example: find the curl of the following three vector fields:
$(a)\ \vec{v} = x\,\hat{y}\ (b)\ \vec{v} = -y\,\hat{x} + x\hat{y}$

$$(a)\ \vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & x & 0 \end{vmatrix} = (0)\hat{x} + (0)\hat{y} + (1)\hat{z} = \hat{z}$$

$$(b)\ \vec{\nabla}\times\vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = (0)\hat{x} + (0)\hat{y} + (1+1)\hat{z} = 2\,\hat{z}$$

Product Rules

$$(a)\ \vec{\nabla}(fg) = f\vec{\nabla}g + g\vec{\nabla}f$$

$$(b)\ \vec{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (B\cdot\vec{\nabla})\vec{A}$$

$$(c)\ \vec{\nabla}\cdot(f\vec{A}) = \vec{A}\cdot\vec{\nabla}f + f(\vec{\nabla}\cdot\vec{A})$$

$$(d)\ \vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

$$(e)\ \vec{\nabla}\times(f\vec{A}) = f(\vec{\nabla}\times\vec{A}) - \vec{A}\times\vec{\nabla}f$$

$$(d)\ \vec{\nabla}\times(\vec{A}\times\vec{B}) = (B\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A})$$

Quotient rules are easily obtained from the above rules

Second Derivatives

(a) $Divergence\ of\ gradient$: $\vec{\nabla} \cdot \vec{\nabla} f = \nabla^2 \mathrm{f} = \Delta \mathrm{f}$ (Laplacian of f)

(b) $Curl\ of\ graidents$: $\vec{\nabla} \times \vec{\nabla} f = 0$

(c) $Gradient\ of\ diveregence$: $\vec{\nabla}(\vec{\nabla} \cdot \vec{v})$

(d) $Divergence\ of\ curl$: $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{v}) = 0$

(e) $\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{v}\right) - \nabla^2 \vec{v}$