Exercises :

Q57 : let a point be selected from the sample space $\mathcal{C} = \{c : 0 < c < 10\}$. let $C \subset \mathcal{C}$ and let the probability set function be $P(C) = \int_C \frac{1}{10} \, dz$. Define the Random variable X to be $X(c) = c^2$. Find the distribution and the p.d.f of X.

→ The c.d.f is defined as :

$$F(x) = Pr(X \leq x) = Pr(C^2 \leq x)$$

$$= Pr(C \leq \sqrt{x})$$

$$= \int_0^{\sqrt{x}} \frac{1}{10} \, dz$$

$$= \frac{1}{10} x \Big|_0^{\sqrt{x}} = \frac{1}{10} \sqrt{x}$$

The pdf :

$$f(x) = \begin{cases} \frac{1}{10} , & 0 < x < 10 \\ 0 , & \text{elsewhere.} \end{cases}$$

$f(x) \geq 0 \quad , \forall x \in \mathbb{R}$ ✓

$\int_0^{10} f(x) dx = \frac{1}{10}(10 - 0) = 1$ ✓

1.58: let the prob. set function $P(A)$ of the random variable X be $P(A) = \int_A f(x)\,dx$

Where: $f(x) = \frac{2x}{9}$, $x \in \mathcal{A} = \{x : 0 < x < 3\}$, let $A_1 = \{x : 0 < x < 1\}$

$A_2 = \{x : 2 < x < 3\}$, compute:

① $P(A_1) = P_r[X \in A_1]$

$$= \int_0^1 \frac{2x}{9}\,dx = \frac{2}{9}\int_0^1 x\,dx = \frac{2}{9}\left(\frac{x^2}{2}\Big|_0^1\right) = \frac{2}{9} \times \frac{1}{2} = \frac{1}{9}$$

② $P(A_2) = P_r(X \in A_2) = \int_2^3 \frac{2x}{9}\,dx = \frac{2}{9}\int_2^3 x\,dx = \frac{2}{9}\left(\frac{x^2}{2}\Big|_2^3\right)$

$$= \frac{2}{9}\left(\frac{9}{2} - \frac{4}{2}\right) = \frac{2}{9} \times \frac{5}{2} = \frac{5}{9}$$

③ $P(A_1 \cup A_2) = P_r(X \in A_1 \cup A_2) = P(A_1) + P(A_2) = \frac{1}{9} + \frac{5}{9} = \frac{6}{9}$ ✓

Q62: Given $\int_A \left(\frac{1}{\pi}(1+x^2)\right) dx$, wher $A \subset \mathcal{A} = \{x : -\infty < x < \infty\}$, show that the integral could serve as a probability set function of a random variable X whose space is $\mathcal{A}$.

Q63: $P(A) = \int_A e^{-x}\,dx$ where $A = \{x : 0 < x < \infty\}$, let $A_K = \{x : 2 - \frac{1}{K} < x \leq 3\}$

$K = 1, 2, \dots$

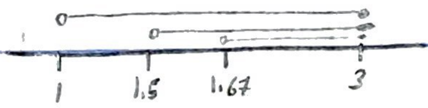
Find $\lim_{K\to\infty} A_K$ and $P\left(\lim_{K\to\infty} A_K\right)$. Find $P(A_K)$ and $\underline{\lim_{K\to\infty} P(A_K)} = P\left(\lim_{K\to\infty} A_K\right)$.

$A_1 = (1, 3]$

$A_2 = (1.5, 3]$

$A_3 = (1.67, 3]$

$\vdots$



set كبيرة وعم تصغر ∴ تقاطع

$\Rightarrow A_K = A_1 \cap A_2 \cap A_3 \cap \dots$

so $\lim_{K\to\infty} A_K = \{x : x = 3\}$

$\rightarrow P\left(\lim_{K\to\infty} A_K\right) = P(x = 3)$

?

lim of sets :

- اذا كانت set صغيرة وقاعدة بتكبر ف lim بتكون الاتحاد
- اذا كانت set كبيرة وقاعدة بتصغر ف lim بتكون التقاطع

Q4 : For each of the following prob. density functions of X, compute $Pr(|X|<1)$ and $Pr(\sqrt{X^2}<9) \rightarrow Pr(-3<X<3)$

a. $f(x) = \begin{cases} \frac{x^2}{18} & , -3<x<3 \\ 0 & , \text{elsewhere} \end{cases}$ cont.

$\rightarrow Pr(|X|<1) = Pr(-1<X<1) = \frac{1}{18}\int_{-1}^{1} x^2\,dx = \frac{1}{18}\left(\frac{x^3}{3}\Big|_{-1}^{1}\right) = \frac{1}{18}\left(\frac{1}{3}+\frac{1}{3}\right)$

$= \frac{1}{\cancel{18}_9} \times \frac{\cancel{2}}{3}$

$= \frac{1}{27}$

$\rightarrow Pr(x^2<9) = Pr(-3<x<3) = \frac{1}{18}\int_{-3}^{3} x^2\,dx = \frac{1}{18}\left(\frac{x^3}{3}\Big|_{-3}^{3}\right)$

$= \frac{1}{18}\left(\frac{27}{3}+\frac{27}{3}\right) = \frac{1}{\cancel{18}_1} \times \frac{\cancel{54}^3}{3} = \frac{3}{3} = 1$

---

b. $f(x) = \begin{cases} \frac{x+2}{18} & , -2<x<4 \\ 0 & , \text{elsewhere} \end{cases}$

$\rightarrow P(-1<x<1) = \frac{1}{18}\int_{-1}^{1} x+2\,dx = \frac{1}{18}\left(\frac{x^2}{2}+2x\Big|_{-1}^{1}\right) = \frac{1}{18}\left(\frac{1}{2}+2-\frac{1}{2}+2\right)$

$= \frac{2}{9}$

$\rightarrow P(-3<x<3) = \frac{1}{18}\int_{-2}^{3} x+2\,dx = \frac{1}{18}\left(\frac{x^2}{2}+2x\Big|_{-2}^{3}\right) = \frac{1}{18}\left(\frac{9}{2}+6-\frac{4}{2}+4\right)$

$= \frac{7}{12}$

Q16:

a. $f(x) = \begin{cases} \left(\frac{1}{2}\right)^x , & x = 1, 2, 3, \ldots \\ 0 , & \text{elsewhere} \end{cases}$

$\left(\frac{1}{2}\right)^x$ is decreasing and hence maximized at $x = 1$

b. $f(x) = \begin{cases} 12x^2(1-x) , & 0 < x < 1 \\ 0 , & \text{elsewhere} \end{cases}$

$f'(x) = 24x(1-x) - 12x^2$ which is maximized at $x = \frac{3}{2}$.

c. $f(x) = \begin{cases} \frac{1}{2}x^2 e^{-x} , & 0 < x < \infty \\ 0 , & \text{elsewhere} \end{cases}$

$f'(x) = xe^{-x} - \frac{1}{2}x^2 e^{-x}$ which is maximized at $x = 2$.

Q68: let $0 < p < 1$. A $(100p)$th percentile (quantile of order p) of the distribution of a random variable X is a value ξ_p s.t $Pr(X < \xi_p) \le p$ and $Pr(X \le \xi_p) \ge p$. Find the twentieth (20^{th}) percentile of the distribution that has p.d.f of $f(x) = \begin{cases} 4x^3 & , 0 < x < 1 \\ 0 & , \text{elsewhere} \end{cases}$.

Hint: with a cont. type r.v X, $Pr(X < \xi_p) = Pr(X \le \xi_p)$ and hence that common value must equal p.

$$\int_{-\infty}^{\xi_{0.2}} f(x)\,dx = 0.2$$

$$\Rightarrow \int_{0}^{\xi_{0.2}} 4x^3\,dx = 0.2 \quad , \quad 0 < \xi_{0.2} < 1$$

$$\Rightarrow x^4 \Big|_0^{\xi_{0.2}} = 0.2$$

$$\Rightarrow (\xi_{0.2})^4 - 0 = 0.2$$

$$\Rightarrow \xi_{0.2} = \sqrt[4]{0.2}$$

$$= 0.05 \; .$$

Q69: Find the c.d.f, with each of the following p.d.f :

a. $f(x) = \begin{cases} 3(1-x^2) & , 0<x<1 \\ 0 & , \text{elsewhere.} \end{cases}$

$$F(x) = \begin{cases} 0 & , x<0 \\ 1-(1-x)^3 & , 0\leq x<1 \\ 1 & , x\geq 1 \end{cases}$$

$0<x<1$: $F(x) = \int_0^x 3(1-w^2)\,dw$ — $1-w=t$, $-dw=dt$

$= -3\int_1^{1-x} t^2\,dt$

$= 3\int_{1-x}^{1} t^2\,dt = 1-(1-x)^3$

b. $f(x) = \begin{cases} \frac{1}{x^2} & , 1<x<\infty \\ 0 & , \text{elsewhere.} \end{cases}$

$x>1$: $F(x) = \int_1^x \frac{1}{w^2}\,dw = \frac{-1}{w^3}\Big|_1^x$

$= \frac{-1}{x^3} - 1$

$$F(x) = \begin{cases} 1-\frac{1}{x^3} & , x\geq 1 \\ 0 & , x<1 \end{cases}$$

c. $f(x) = \begin{cases} \frac{1}{3} & , 0<x<1 \text{ or } 2<x<4 \\ 0 & , \text{elsewhere} \end{cases}$

$0<x<1$: $F(x) = \int_0^x \frac{1}{3}\,dw = \frac{x}{3}$

$2<x<4$: $F(x) = \frac{1}{3} + \int_2^x \frac{1}{3}\,dw = \frac{1}{3} + \frac{w}{3}\Big|_2^x$

$= \frac{x}{3} - \frac{1}{3}$

$$F(x) = \begin{cases} 0 & , x<0 \\ \frac{x}{3} & , 0\leq x<1 \\ \frac{1}{3} & , 1\leq x<2 \\ \frac{x}{3}-\frac{1}{3} & , 2\leq x<4 \\ 1 & , 4\leq x \end{cases}$$