

Chapter = 4 =

Sampling of Continuous-Time Signals

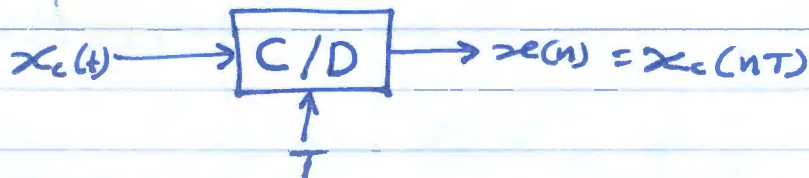
Periodic Sampling:

$$x(n) = x_c(nT), \quad -\infty < n < \infty$$

$T \rightarrow$ sampling period

$$f_s = \frac{1}{T} \rightarrow \text{Sampling Frequency (samples/sec)}$$

$$\Omega_s = \frac{2\pi}{T} \text{ sampling frequency (radians/sec)}$$

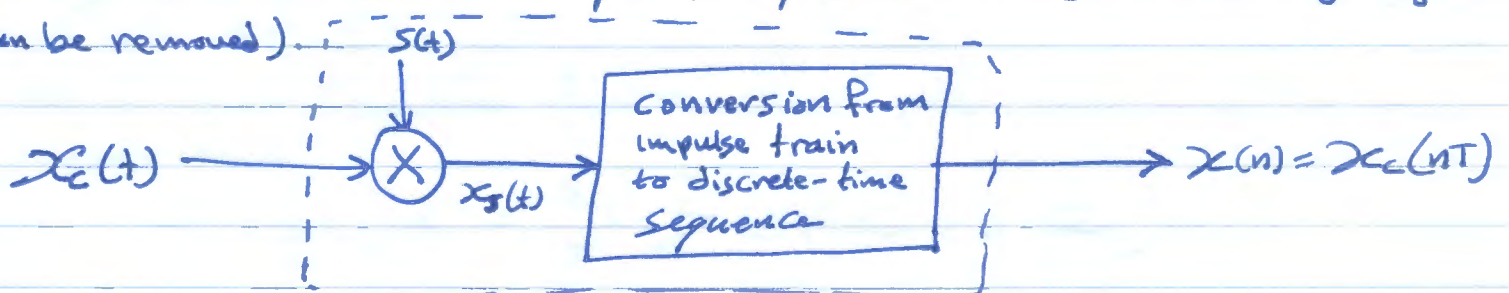


Ideal Continuous-to-Discrete-time Converter (C/D).

* In practical, Sampling is implemented by Analog-to-Digital (A/D) Converter.

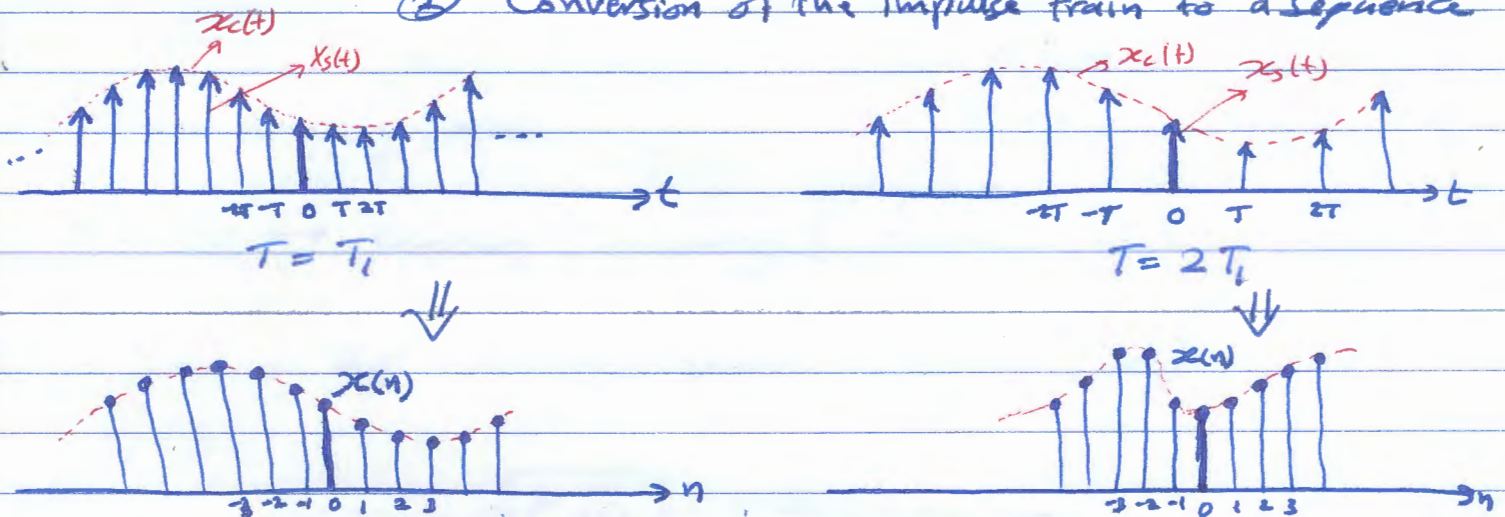
- * A/D - quantization of output samples?
- Linearity of quantization steps?
- Need for sampling-and-hold circuits.
- Limitations on the sampling rate.

* Sampling is not Invertible. i.e. we cannot re-generate $x_c(t)$ from $x(n)$. Since many Continuous-time signals can have the same output samples $x(n)$. (This ambiguity can be removed).



Two Stages:

- Sampling → ① Impulse Train Modulator (followed by)
② Conversion of the impulse train to a sequence



Frequency-Domain Representation of Sampling:

$$S(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \quad \delta(t) \rightarrow \text{Unit Impulse Function or Dirac Delta Function}$$

$$x_s(t) = x_c(t) S(t) \\ = x_c(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

* By Shifting Property of Impulse Function, $x_s(t)$ can be expressed as

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT) \delta(t - nT)$$

$F\{x_s(t)\}$?

Since $x_s(t) = x_c(t) \cdot S(t)$ (Multiplication in Time-domain).

$\Rightarrow F\{x_s(t)\}$ is convolution of Fourier Transform $X_c(j\omega)$ and $S(j\omega)$.

* Fourier Transform of a periodic impulse train is a periodic impulse train:

$$S(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$\Omega_s = \frac{2\pi}{T} \rightarrow$ Sampling Freq. in radians per second.

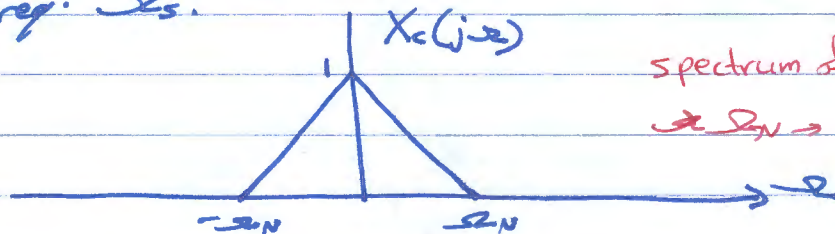
$$X_s(j\Omega) = \frac{1}{2\pi} X_c(j\Omega) * S(j\Omega)$$

↑ Continuous Convolution

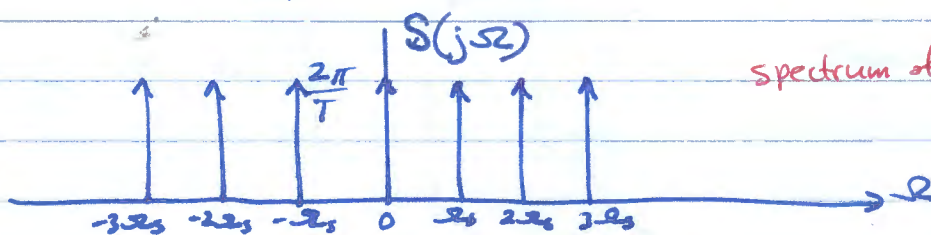
$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

↑ $\mathcal{F}\{x_c(t)\}$
periodically repeated copies of $X_c(j\Omega)$

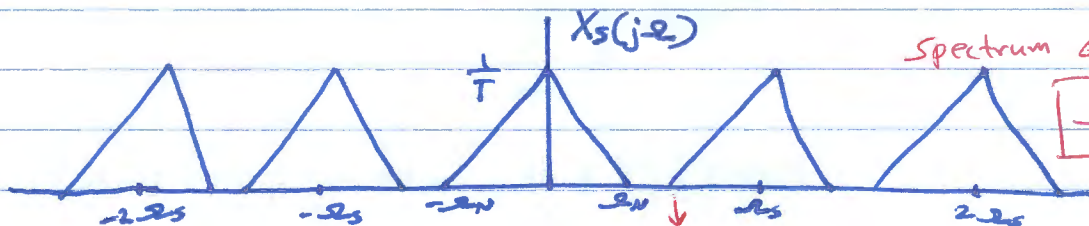
* Copies of $X_c(j\Omega)$ are shifted by Integer multiples of Sampling Freq. Ω_s .



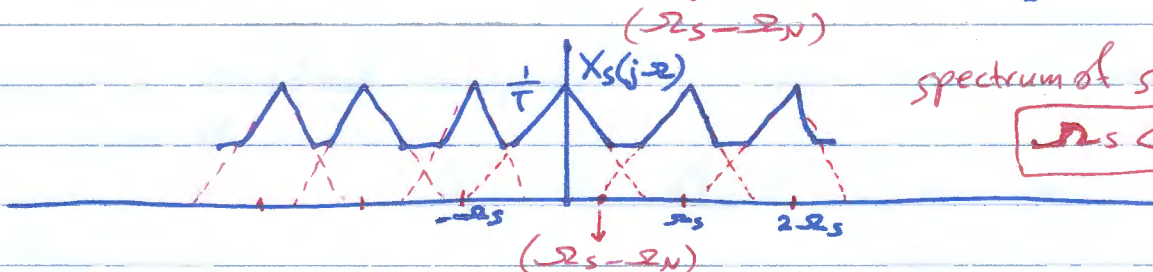
spectrum of original signal
 $\Omega_N \rightarrow$ highest freq. in $x_c(t)$.



spectrum of Sampling Function

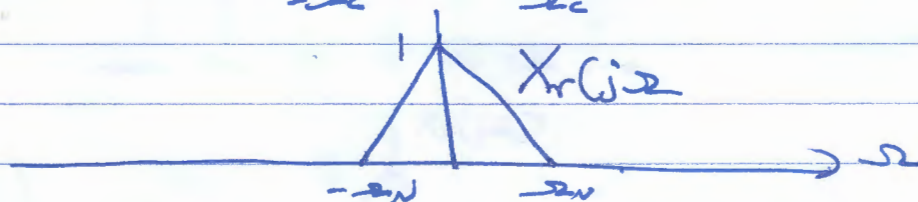
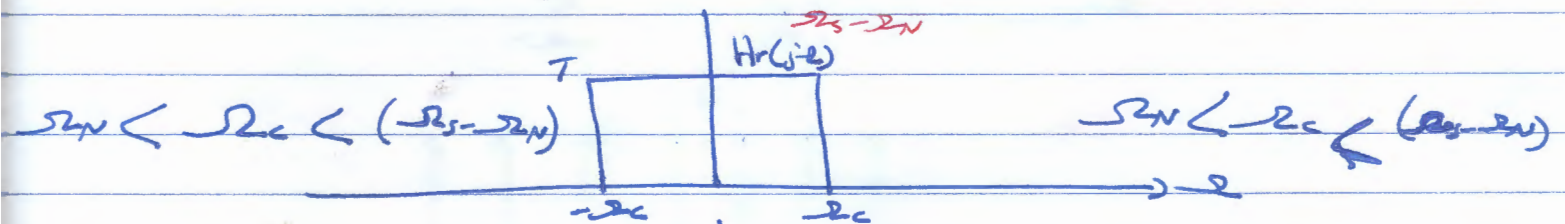
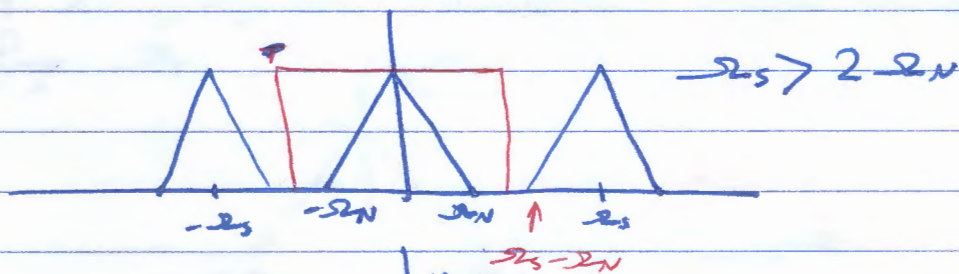
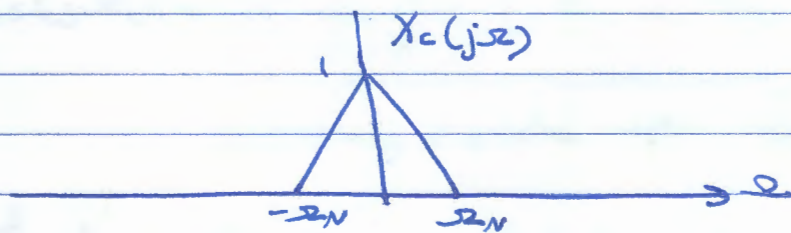
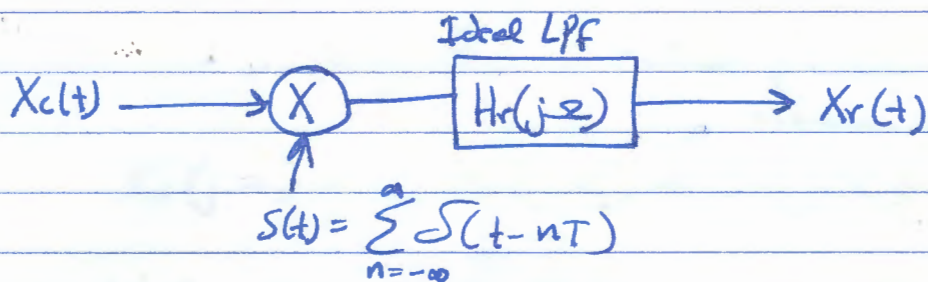


spectrum of Sampled Signal
 $\Omega_s > 2\Omega_N$



spectrum of sampled signal
 $\Omega_s < 2\Omega_N$

When, $\Omega_s - \Omega_N > \Omega_N \Rightarrow$ or $\Omega_s > 2\Omega_N$
 $\Rightarrow$ replicas of $X_c(j\Omega)$ don't ~~overlap~~ overlap $\Rightarrow x_c(t)$
 Can be recovered from $x_s(t)$ with an ideal lowpass filter as shown:



$$X_r(j\omega) = H_r(j\omega) X_s(j\omega)$$

$H_r(j\omega) \rightarrow$ ideal lowpass filter with gain T and cutoff freq. ω_c such that:

$$\omega_N < \omega_c < (\omega_s - \omega_N)$$

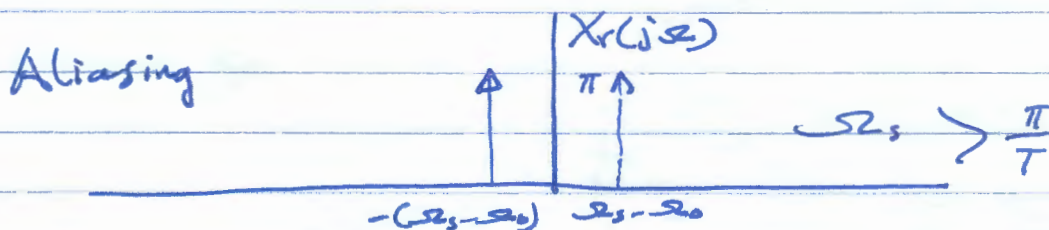
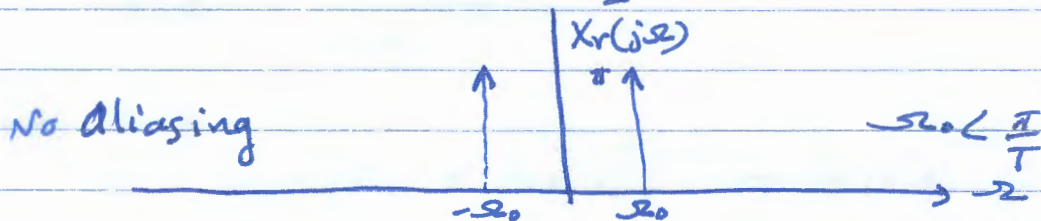
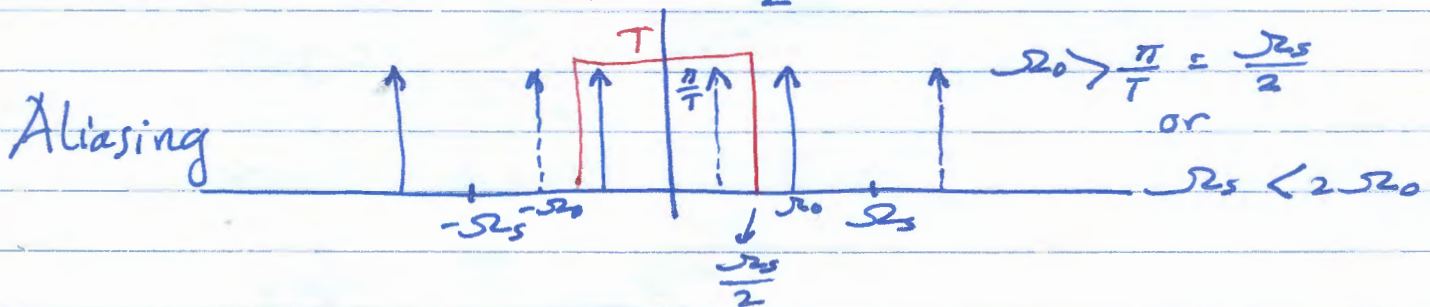
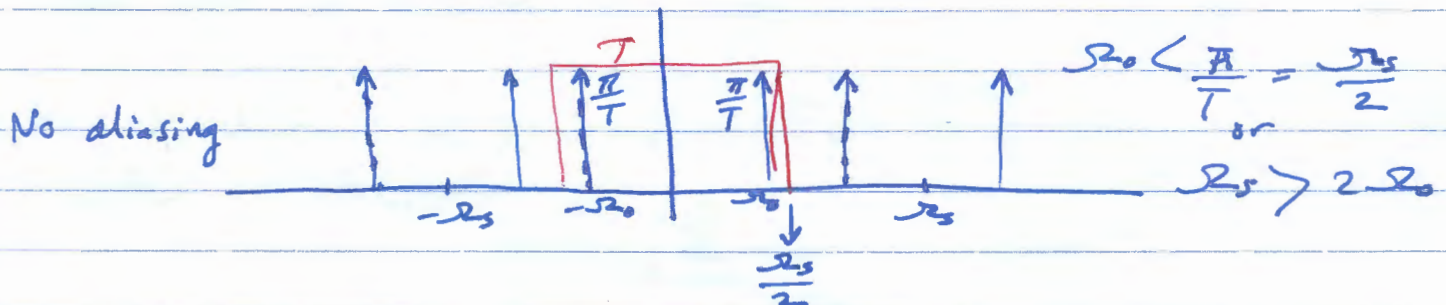
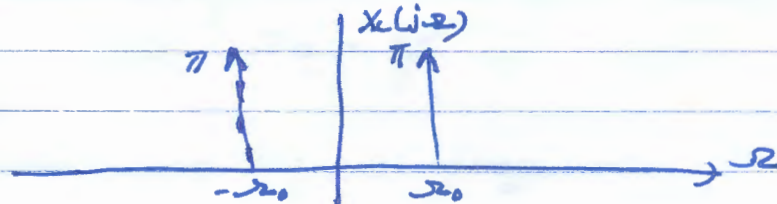
Then,

$$X_r(j\omega) = X_c(j\omega) \quad \text{and} \quad x_r(t) = x_c(t).$$

* If $\omega_s \leq 2\omega_N$, copies of $X_c(j\omega)$ overlap, so that when they are added together, $X_c(j\omega)$ is no longer recoverable by lowpass filter.

* In this case, The reconstructed output $x_r(t)$ is related to the input $x_c(t)$ through a distortion referred to as aliasing.

* Assume $x_c(t) = \cos \omega_c t$



* With No aliasing: $x_r(t) = \cos \omega_c t$

* With aliasing: $x_r(t) = \cos(\omega_s - \omega_c)t$

I.e. Higher freq. signal $\cos \omega_c t$ has taken on the identity (alias) of lower freq. signal $\cos(\omega_s - \omega_c)t$.

Nyquist Sampling Theorem

Let $x_c(t)$ be a bandlimited signal with $X_c(j\Omega) = 0$ for $|\Omega| \geq \Omega_N$.
Then, $x_c(t)$ is uniquely determined by its samples $x(n) = x_c(nT)$
 $n = 0, \pm 1, \pm 2, \dots$ if $\Omega_s = \frac{2\pi}{T} \geq 2\Omega_N$

$\Omega_N \rightarrow$ referred to as Nyquist Frequency.

$2\Omega_N \rightarrow$ " " " " " rate.

* our objective is to express $X(e^{j\omega}) = \mathcal{F}\{x(n)\}$ in term of $X_s(j\Omega)$ and $X_c(j\Omega)$.

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_c(nT) \delta(t - nT)$$

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} X_c(nT) e^{-j\Omega nT}$$

Since, $x(n) = x_c(nT)$

and

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

It follows that:

$$X_s(j\Omega) = X(e^{j\omega}) \Big|_{\omega = \Omega T} = X(e^{j\Omega T})$$

From previous eq.

$$X_s(j\Omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\Omega - k\Omega_s))$$

so,

$$X(e^{j\Omega T}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\frac{\Omega}{T} - \frac{2\pi k}{T}))$$

or equivalently,

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\frac{\omega}{T} - \frac{2\pi k}{T}))$$

* As we see $X(e^{j\omega})$ is frequency-scaled version of $X_s(j\Omega)$ with freq. scaling specified by $\omega = \Omega T$.

* This frequency scaling can be thought as normalization of freq. axis, so $\Omega = \Omega_s$ in $X_s(j\Omega)$ is normalized to $\omega_s = 2\pi$ for $X(e^{j\omega})$.

* This is due to time normalization in transformation from $x_s(t)$ to $x(n)$. $x_s(t)$ retains spacing between samples equal to sampling period T . In contrast, the "spacing" of sequence values $x(n)$ is always unity i.e. time axis is normalized by a factor of T . Correspondingly, in frequency domain, freq-axis is normalized by a factor of $\frac{1}{T}$.

Example: $x_c(t) = \cos(4000\pi t)$, $T = \frac{1}{6000}$ sec.

$$x(n) = x_c(nT) = \cos(4000\pi Tn) = \cos \omega_0 n, \text{ where}$$

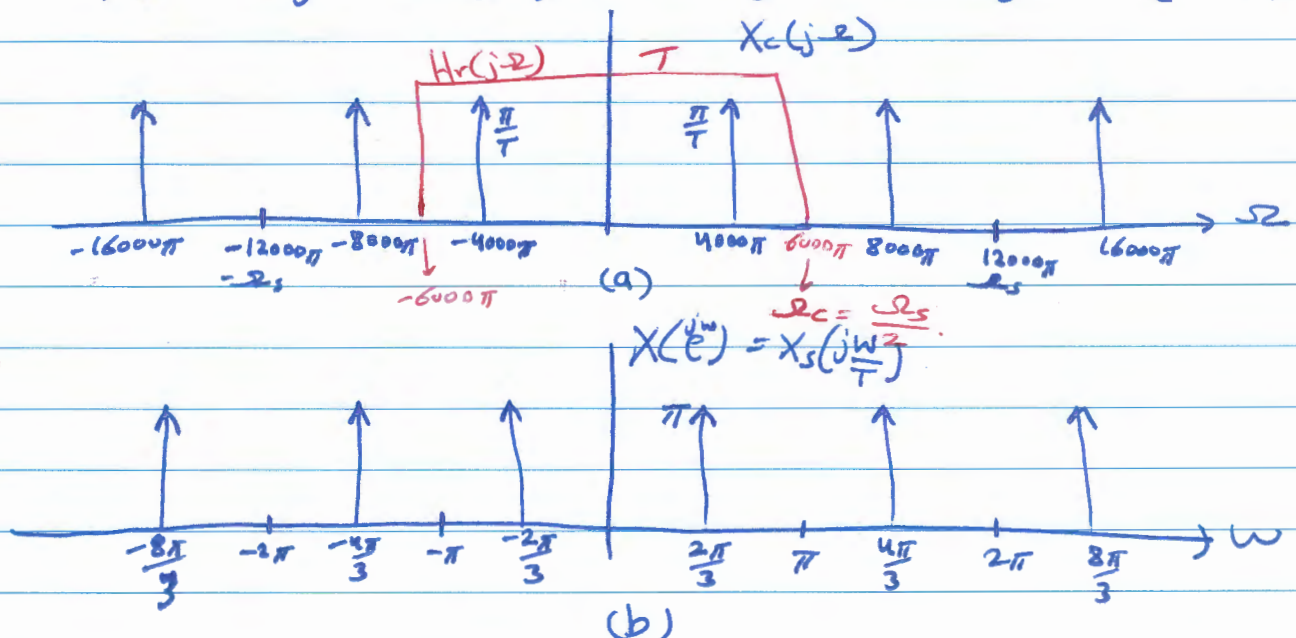
$$\omega_0 = 4000\pi T = \frac{2\pi}{3}.$$

In this case,

$$\Omega_s = \frac{2\pi}{T} = 12000\pi, \text{ and highest freq. of } x_c(t) \text{ is}$$

$\Omega_0 = 4000\pi$, so conditions of Nyquist sampling theorem are satisfied. and there is no aliasing.

$$F\{x_c(t)\} = X_c(j\Omega) = \pi \delta(\Omega - 4000\pi) + \pi \delta(\Omega + 4000\pi)$$



* Fig. (a) shows:

$$X_s(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\omega - k\omega_s))$$

For $\omega_s = 12000\pi \Rightarrow X_c(j\omega)$ is a pair of impulses at $\omega = \pm 4000\pi$ and we shifted copies of $X_c(j\omega)$ centred at $\pm\omega_s, \pm 2\omega_s, \dots$ etc.

* Fig. (b) is a plot of $X(e^{j\omega}) = X_s(j\frac{\omega}{T})$ as a function of normalized freq. $\omega = \omega_s T = \frac{\omega}{f_s}$ (Scaling the independent variable of an impulse also scales its area, i.e. $\delta(\frac{\omega}{T}) = T\delta(\omega)$).

* Original freq. $\omega_0 = 4000\pi \Rightarrow \omega_0 = 4000\pi T = \frac{2\pi}{3}$ which satisfies $\omega_0 < \pi \Leftrightarrow \omega_0 = 4000\pi < \frac{\pi}{T} = 6000\pi$.

Fig. (a) also shows Ideal reconstruction filter $H_r(j\omega)$ for given sampling frequency of $\omega_s = 12000\pi$.

Example: Suppose $x_c(t) = \cos(16000\pi t)$, Sampling period $T = \frac{1}{6000}$

This sampling period T fails to satisfy the Nyquist Criterion, since $\omega_s = \frac{2\pi}{T} = 12000\pi < 2\omega_0 = 32000\pi$.
 $\Rightarrow$ we expect to see aliasing !!

$X_s(j\omega)$ for this case is identical to that of the previous example, but now the impulse located at $\omega = -4000\pi$ is from $X_c(j(\omega - \omega_s))$ rather than from $X_c(j\omega)$, and impulse at $\omega = 4000\pi$ is from $X_c(j(\omega + \omega_s))$.

* plotting $X(e^{j\omega}) = X_s(j\frac{\omega}{T})$ as a function of ω yields the same graph in the previous example Fig. (b). Since we are normalizing by the same sampling period.

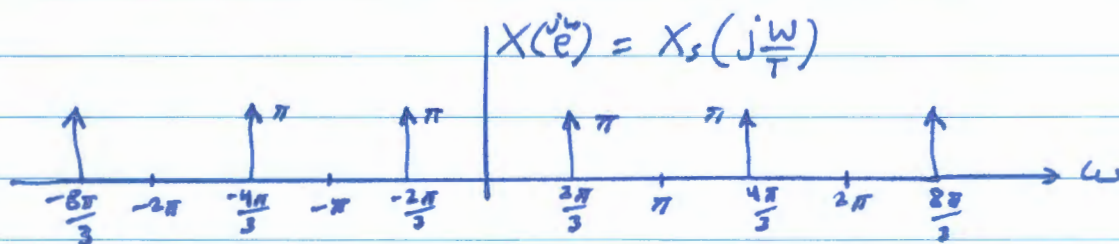
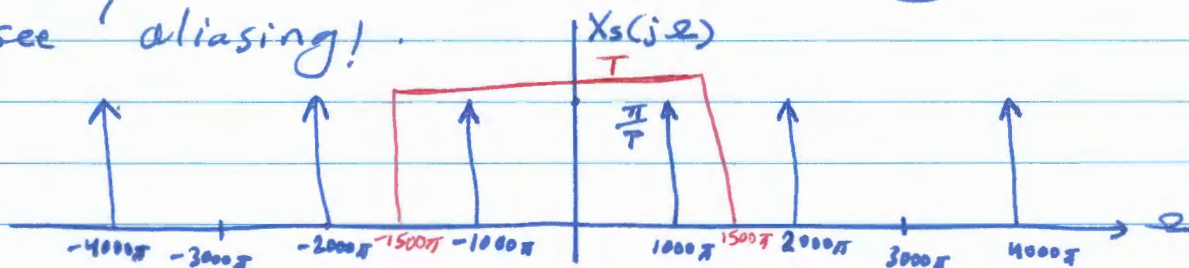
* Reason for this: sequence of samples, $x(n)$, in both cases is the same. $\cos(16000\pi n/6000) = \cos(2\pi n + 4000\pi n/6000) = \cos(\frac{2\pi}{3}n)$.

* (Note we can add any integer multiple of 2π to argument of cosine with changing its value).

* So, we got the same seq. $x(n) = \cos(\frac{2\pi}{3}n)$ by sampling two different continuous-time signals with the same sampling freq.

* In the second case, output of filter $H_r(j\Omega)$ is $\cos(4000\pi t)$ which is not the original signal $x_c(t) = \cos(16000\pi t)$.

Example: suppose $x_c(t) = \cos(4000\pi t) \Rightarrow \Omega_0 = 4000\pi$
 $T = \frac{1}{1500} \Rightarrow$ fails to satisfy Nyquist rate, since
 $\Omega_s = \frac{2\pi}{T} = 3000\pi < 2\Omega_0 = 8000\pi \Rightarrow$ so we expect to see $\frac{1}{T}$ aliasing!



$\Omega = 1000\pi$ comes from $X_c(j(-\Omega + \Omega_s))$ and $-1000\pi \rightarrow$ comes from $X_c(j(-\Omega - \Omega_s))$.

$X(e^{j\omega})$ plot $\Rightarrow$ corresponds to seq. $x(n) = \cos(\frac{2\pi}{3}n)$

* Same discrete-time signal ($x(n)$) may result from sampling the same continuous-time signal at different sampling rate if of those sampling rates fails to satisfy sampling theorem.

* Re-constructed signal from $H_r(j\omega)$ is $\cos(1000\pi t)$ and $\cos(4000\pi t)$.

Sec. 4.3 Reconstruction of Bandlimited signal from its samples.

If $x(n)$ is given $\Rightarrow$ we can form an impulse train $x_s(t)$ in which successive impulses are assigned an area equal to sequence value i.e.

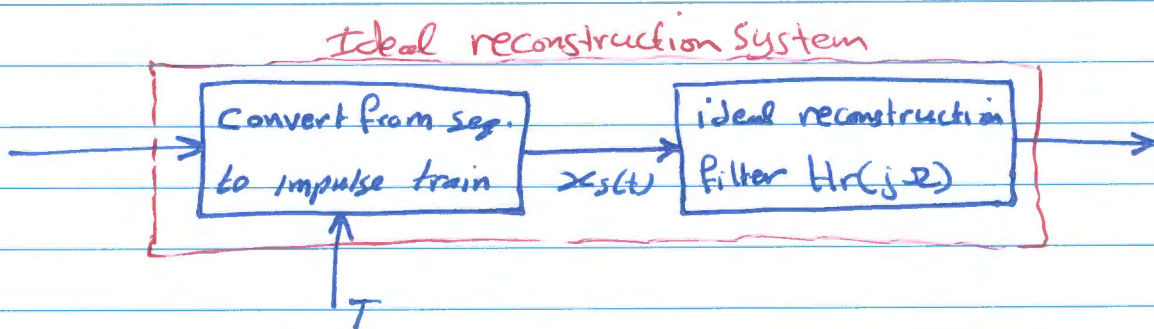
$$x_s(t) = \sum_{n=-\infty}^{\infty} x(n) \delta(t-nT)$$

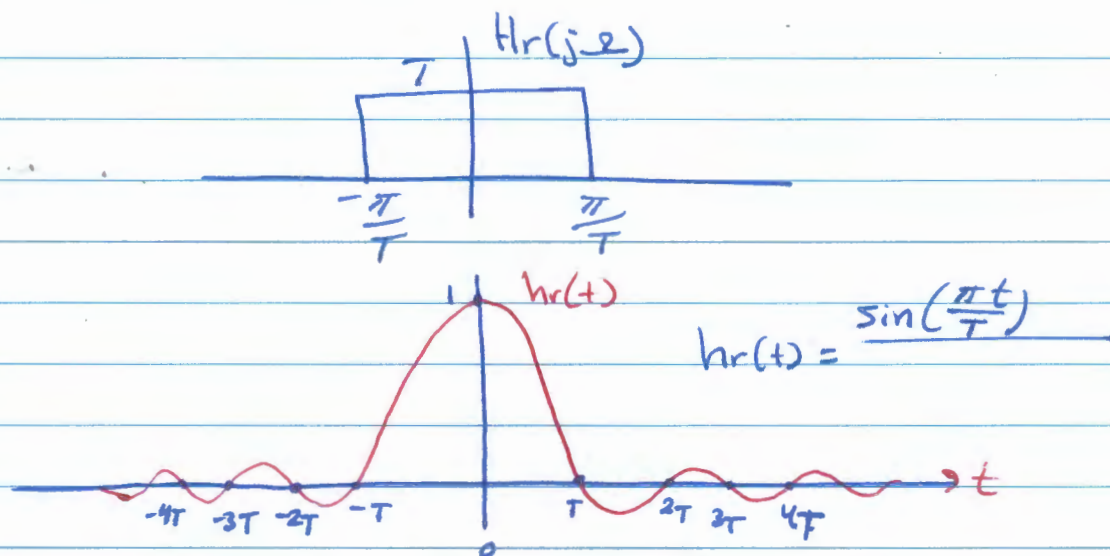
The n^{th} sample is associated with impulse at $t=nT$, where T is sampling period associated with $x(n)$.



$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n) h_r(t-nT)$$

* Recall ideal reconstruction filter has a gain of T (to compensate for a factor $\frac{1}{T}$) and cutoff freq. ω_c between ω_N and $\omega_S - \omega_N$.





Common choices for $\omega_c = \frac{\omega_s}{2} = \frac{\pi}{T}$ (No aliasing as long as $\omega_s > 2\omega_m$).

So,

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

* As we know if $x(n) = x_c(nT)$, where $X_c(j\omega) = 0$ for $|\omega| > \frac{\pi}{T}$ then $x_r(t)$ is equal to $x_c(t)$.

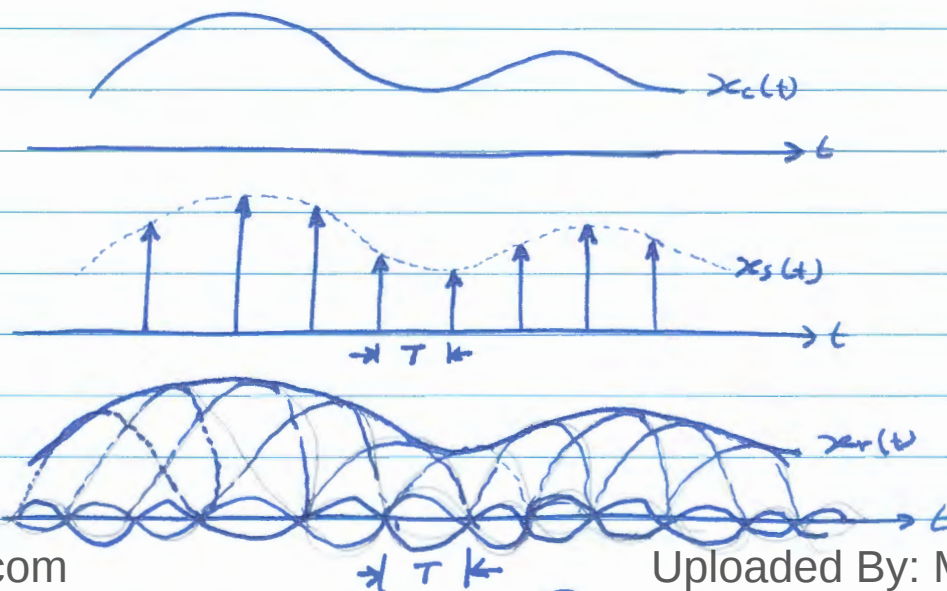
More illustration:

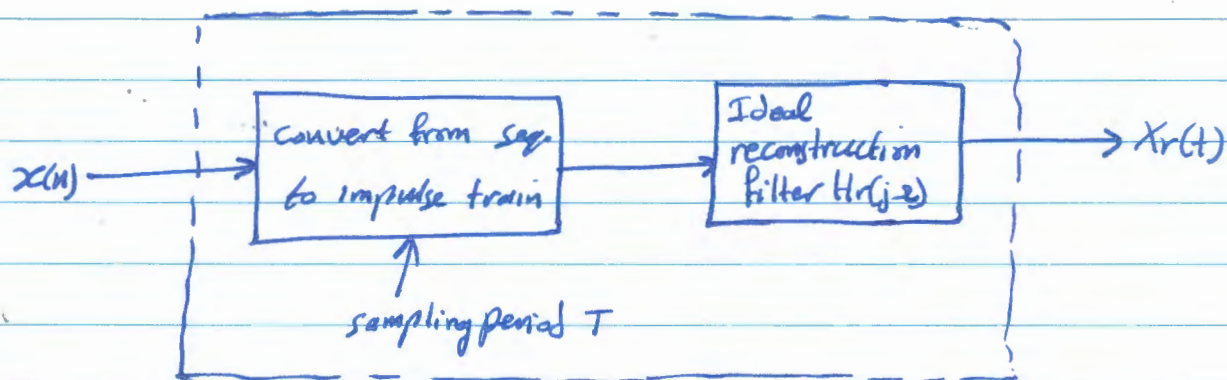
by looking at $h_r(t)$ above, we note $h_r(0) = 1$

$h_r(nT) = 0$ for $n = \pm 1, \pm 2, \dots$

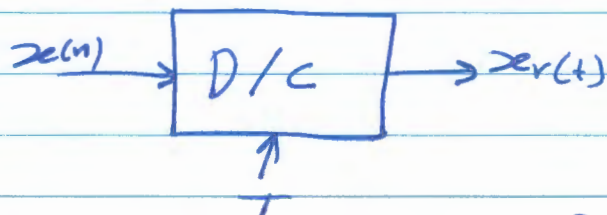
So, that $\Rightarrow$ if $x(n) = x_c(nT)$, then $x_r(mT) = x_c(mT)$ for all integer m .

$\Rightarrow$ That is, the reconstructed signal has the same values at sampling times as the original continuous-time signal, independently of T .





Ideal discrete-to-continuous time Converter.



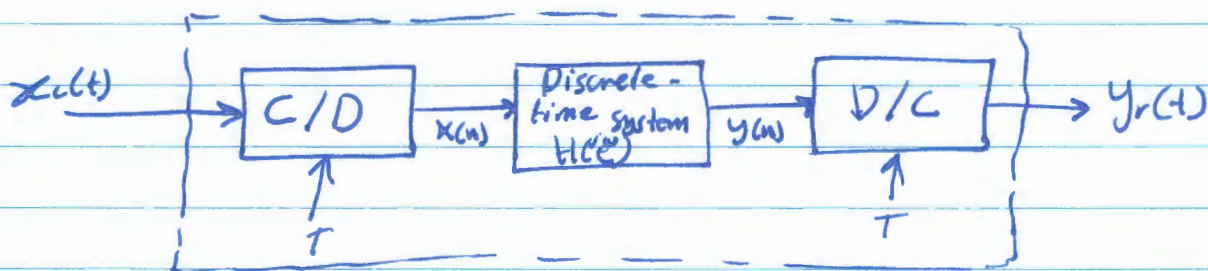
* relationship between input/output in Frequency-domain

$$X_r(j\Omega) = \sum_{n=-\infty}^{\infty} x(n) H_r(j\Omega) e^{-j\Omega n T}$$

$$X_r(j\Omega) = H_r(j\Omega) X(e^{j\Omega T})$$

normalized freq. ω is replaced by ΩT .

Sec 4.4 Discrete-time processing of continuous-time signals



* We assume sampling period of C/D and D/C are the same (but it is not necessary always)

C/D → produces $x(n) = x_c(nT)$

$$X(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\frac{\omega}{T} - \frac{2\pi k}{T})) \quad \text{--- (*)}$$

D/C → creates a continuous-time output

$$y_r(t) = \sum_{n=-\infty}^{\infty} y(n) \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$

$$\begin{aligned} Y_r(j\omega) &= H_r(j\omega) Y(e^{j\omega T}) \\ &= \begin{cases} T X(e^{j\omega T}), & |\omega| < \frac{\pi}{T} \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

LTI systems:

$$Y(e^{j\omega}) = \underbrace{H(e^{j\omega})}_{\text{freq. response of LTI system}} X(e^{j\omega})$$

So,

$$Y_r(j\omega) = H_r(j\omega) H(e^{j\omega T}) X(e^{j\omega T})$$

Substitute $\omega = \omega T$ in eq. (*) above.

$$Y_r(j\omega) = H_r(j\omega) H(e^{j\omega T}) \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j(\omega - \frac{2\pi k}{T}))$$

* If $X_c(j\omega) = 0$ for $|\omega| \geq \frac{\pi}{T}$, then ideal lowpass reconstruction filter $H_r(j\omega)$ cancels the factor $1/T$ and selects only the term for $k=0$.

$$Y_r(j\omega) = \begin{cases} H(e^{j\omega T}) X_c(j\omega), & |\omega| < \frac{\pi}{T} \\ 0, & |\omega| \geq \frac{\pi}{T} \end{cases}$$

* If $X_c(j\omega)$ is bandlimited and sampling rate is above Nyquist rate the output-input relation:

$$Y_r(j\omega) = H_{\text{eff}}(j\omega) X_c(j\omega)$$

When,

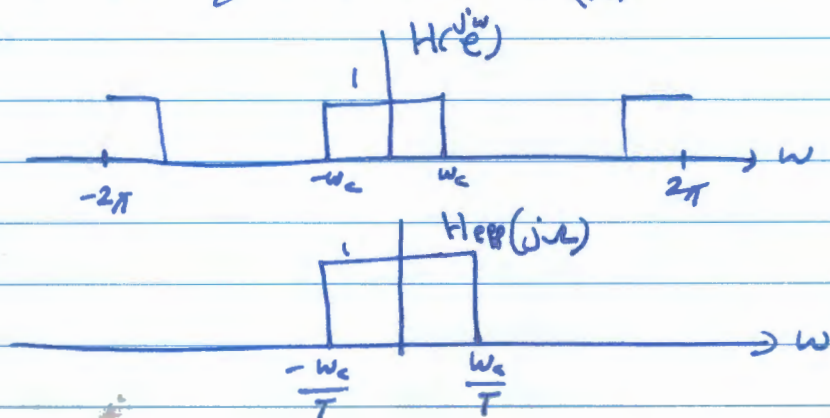
$$H_{\text{eff}}(j\omega) = \begin{cases} H(e^{j\omega T}), & |\omega| < \frac{\pi}{T} \\ 0, & |\omega| \geq \frac{\pi}{T} \end{cases}$$

Two conditions for this relationship:

- ① Discrete-time system must be LTI
- ② Input signal must be bandlimited, and sampling rate must be high enough, so that any aliased components are removed by discrete-time system.

Example:

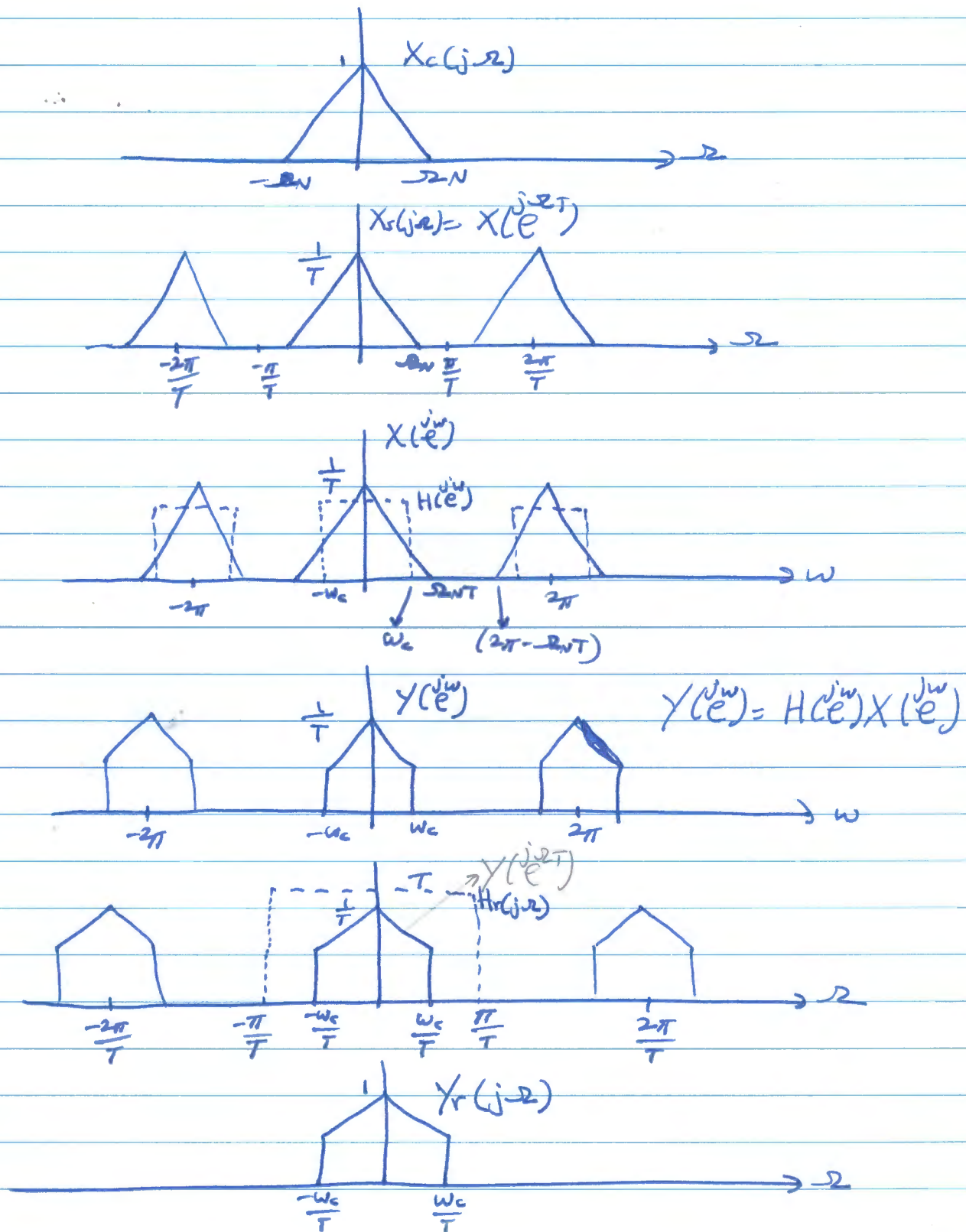
$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases} \rightarrow \text{periodic with period } 2\pi$$



* For bandlimited input sampled above Nyquist rate, overall sys. is LTI Continuous-time system with Freq. response

$$H_{eff}(j\omega) = \begin{cases} 1, & |\omega| < \omega_c \text{ or } |\omega| < \frac{\omega_c}{T} \\ 0, & |\omega| > \omega_c \text{ or } |\omega| > \frac{\omega_c}{T} \end{cases}$$

The effective freq. response $\rightarrow$ ideal lowpass filter with cutoff freq. $\omega_c = \frac{\omega_c}{T}$.



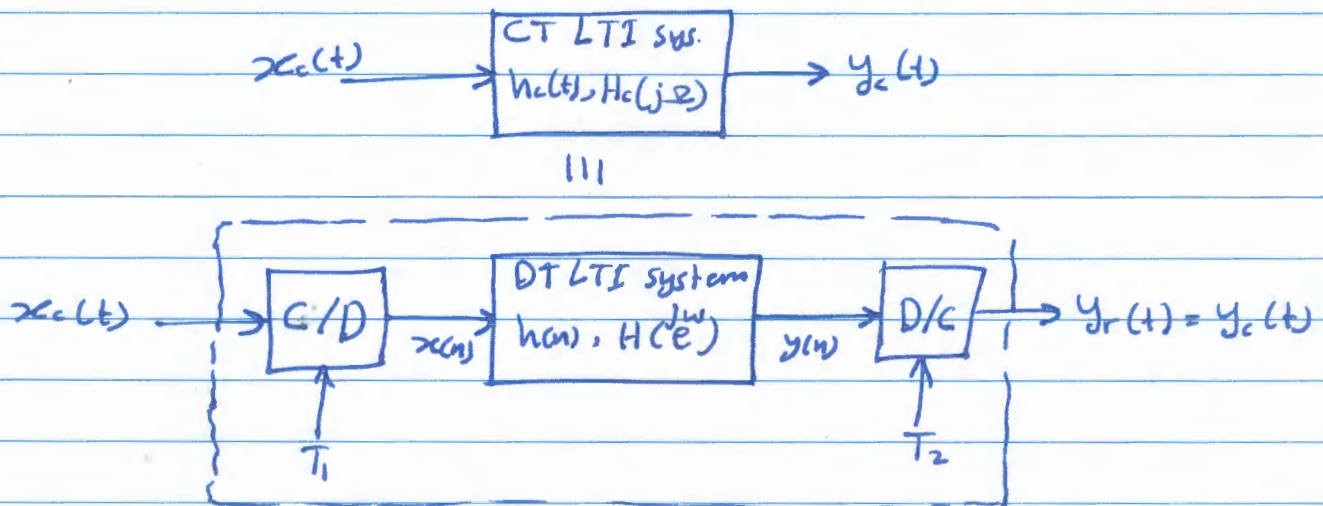
* Important points:

* Low pass discrete-time filter with cutoff freq. ω_c has effect of an ideal lowpass filter with cutoff $\omega_c = \frac{\omega_c}{T}$ when used in the overall system (This cutoff depends on ω_c and T).

* To get the same lowpass filter, we can fix the discrete-time lowpass filter cutoff (ω_c) and vary sampling freq. F_s or T .

E.g. If T were chosen so that $\Omega_c T < \omega_c \Rightarrow y_r(t) = x_c(t)$.
 For no aliasing $\Rightarrow (2\pi - \Omega_c T) > \omega_c$
 compared with Nyquist requirements that
 $(2\pi - \Omega_c T) > \Omega_c T$

Impulse Invariance



$$H_{eff}(j\Omega) = H_c(j\Omega)$$

recall,

$$H_{eff} = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \frac{\pi}{T} \\ 0, & |\Omega| \geq \frac{\pi}{T} \end{cases}$$

* This specifies how to choose $H(e^{j\omega})$ so that $H_{eff}(j\Omega) = H_c(j\Omega)$ i.e.

$$H(e^{j\omega}) = H_c(j\frac{\omega}{T}), \quad |\omega| < \pi \quad \text{--- (1)}$$

and T should be chosen such that,

$$H_c(j\Omega) = 0, \quad |\Omega| \geq \frac{\pi}{T} \quad \text{--- (2)}$$

Under these two constraints (1 and 2), also $h(n) = T h_c(nT)$
 i.e. impulse response of DT system is a scaled, sampled version of $h_c(t)$. proof in page 161 (Textbook).

Example: We want Ideal LPT (Discrete-time) with $\omega_c < \pi$. We can do this by sampling a continuous-time ideal lowpass filter with cutoff frequency $\Omega_c = \frac{\omega_c}{T} < \frac{\pi}{T}$ defined by:

$$H_c(j\Omega) = \begin{cases} 1, & |\Omega| < \Omega_c \\ 0, & |\Omega| \geq \Omega_c \end{cases}$$

$$\Rightarrow h_c(t) = \frac{\sin(\Omega_c t)}{\pi t}$$

So, we define impulse response of DT system to be:

$$h(n) = T h_c(nT) = T \frac{\sin(\Omega_c nT)}{\pi nT} = \frac{\sin(\omega_c n)}{\pi n}$$

Where, $\omega_c = \Omega_c T$

* We have already shown that Fourier Transform of $h(n)$ is

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c \leq |\omega| \leq \pi \end{cases}$$

Which is identical to $H_c(j\frac{\omega}{T})$.