3.1 Matrices

- The matrix A of order $m \times n$ consists of m rows and n columns:

$$A = \begin{array}{cc} \begin{array}{cccccc} \text{Column 1} & \text{Column 2} & \text{Column 3} & & \text{Column } n \end{array} & \\ \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} & \begin{array}{c} \text{Row 1} \\ \text{Row 2} \\ \vdots \\ \text{Row } m \end{array} \end{array}$$

- The numbers in the matrix are called entries or elements
- The matrix is called square if $m=n$
- The transpose of the matrix A is

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ a_{13} & a_{23} & \cdots & a_{m3} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

- The negative of the matrix A is

$$-A = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & -a_{22} & -a_{23} & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{m1} & -a_{m2} & -a_{m3} & \cdots & -a_{mn} \end{bmatrix}$$

- Any matrix with one row is called row matrix or row vector

Exp $A = \begin{bmatrix} 1 & -2 \end{bmatrix}_{1\times 2}$, $B = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix}_{1\times 3}$

- Any matrix with one column is called column matrix or column vector

Exp $C = \begin{bmatrix} \sqrt{5} \\ \pi \end{bmatrix}_{2\times 1}$, $D = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}_{4\times 1}$

- Any matrix whose entries are all zeros is called zero matrix

Exp $Z_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $Z_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

- Two matrices are equal if they have same order and all corresponding entries are equal

Exp $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ and $\begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 0 & 4 \end{bmatrix}$

- If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ have same order then

the sum $A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{bmatrix}$

Exp If $A = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$ then

$$A + B = \begin{bmatrix} 1+5 & 0+-2 \\ 2+3 & -1+1 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 5 & 0 \end{bmatrix} = B + A$$ "addition is commutative"

- If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and c is any scalar (real number), then

the scalar multiplication $cA = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$

Exp If $A = \begin{bmatrix} 2 & 4 \\ 6 & -10 \end{bmatrix}$ then $\frac{1}{2}A = \begin{bmatrix} \frac{1}{2}(2) & \frac{1}{2}(4) \\ \frac{1}{2}(6) & \frac{1}{2}(-10) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$

Exp Given the following matrices

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 4 & 2 \\ 7 & 8 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & -3 \\ 4 & 0 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 4 & 2 \\ 7 & -8 & 4 \end{bmatrix}$$

Find ① the order of A, B, C

$A_{3\times3}$, $B_{2\times3}$, $C_{3\times3}$

② the negative of B

$$-B = \begin{bmatrix} -1 & 1 & 3 \\ -4 & 0 & -2 \end{bmatrix}$$

③ the entries (elements) a_{32} , a_{13} , b_{22} , b_{23} , b_{32} , c_{32}

$a_{32} = 8$, $a_{13} = 3$, $b_{22} = 0$, $b_{23} = 2$, b_{32} undefined , $c_{32} = -8$

④ the transpose of B and C

$$B^T = \begin{bmatrix} 1 & 4 \\ -1 & 0 \\ -3 & 2 \end{bmatrix}, \quad C^T = \begin{bmatrix} 1 & 3 & 7 \\ 0 & 4 & -8 \\ -3 & 2 & 4 \end{bmatrix}$$

⑤ the order of B^T , C^T

$B^T_{3\times2}$, $C^T_{3\times3}$

⑥ Whic matrix is square?

$A_{3\times3}$ and $C_{3\times3}$ are square matrices since $m = n = 3$

⑦ Does $A = C$?

No, since $a_{13} \neq c_{13}$

8) Find $A + C$ "addition"

$$A+C = \begin{bmatrix} 1 & 0 & 3 \\ 3 & 4 & 2 \\ 7 & 8 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -3 \\ 3 & 4 & 2 \\ 7 & -8 & 4 \end{bmatrix} = \begin{bmatrix} 1+1 & 0+0 & 3+^{-}3 \\ 3+3 & 4+4 & 2+2 \\ 7+7 & 8+^{-}8 & 3+4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 6 & 8 & 4 \\ 14 & 0 & 7 \end{bmatrix}$$

9) Find $B + C$

$$B+C = \underset{2\times3}{\begin{bmatrix} 1 & -1 & -3 \\ 4 & 0 & 0 \end{bmatrix}} + \underset{3\times3}{\begin{bmatrix} 1 & 0 & -3 \\ 3 & 4 & 2 \\ 7 & -8 & 4 \end{bmatrix}}$$

not possible since they have different order

10) Find $-2B$ "scalar multiplication"

$$-2B = -2\begin{bmatrix} 1 & -1 & -3 \\ 4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} (-2)(1) & (-2)(-1) & (-2)(-3) \\ (-2)(4) & (-2)(0) & (-2)(0) \end{bmatrix} = \begin{bmatrix} -2 & 2 & 6 \\ -8 & 0 & 0 \end{bmatrix}$$

11) Give zero matrix has order as B

$$Z = \underset{2\times3}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

12) Give zero matrix has order as A

$$Z = \underset{3\times3}{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

13) Find $A^T + C^T$

$$A^T + C^T = \begin{bmatrix} 1 & 3 & 7 \\ 0 & 4 & 8 \\ 3 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 7 \\ 0 & 4 & -8 \\ -3 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 14 \\ 0 & 8 & 0 \\ 0 & 4 & 7 \end{bmatrix} = (A+C)^T$$

Exp Let $A = \begin{bmatrix} 5 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -2 \\ -3 & 5 \end{bmatrix}$

Find $3A^T - 4B$

$$3A^T - 4B = 3\begin{bmatrix} 5 & 1 \\ 3 & 2 \end{bmatrix} - 4\begin{bmatrix} 4 & -2 \\ -3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 3 \\ 9 & 6 \end{bmatrix} - \begin{bmatrix} 16 & -8 \\ -12 & 20 \end{bmatrix} = \begin{bmatrix} 15-16 & 3-^-8 \\ 9-^-12 & 6-20 \end{bmatrix} = \begin{bmatrix} -1 & 11 \\ 21 & -14 \end{bmatrix}$$

Exp Find x, y, z, w if

① $\begin{bmatrix} 2x & 1 & 0 \\ 0 & y+1 & z \\ w & 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 0 \\ 0 & 1 & 1-z \\ 2w & 2 & 1 \end{bmatrix}$

$2x = 6 \Rightarrow \boxed{x = 3}$

$y+1 = 1 \Rightarrow \boxed{y = 0}$

$z = 1-z \Rightarrow 2z = 1 \Rightarrow \boxed{z = \frac{1}{2}}$

$w = 2w \Rightarrow \boxed{w = 0}$

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② $\begin{bmatrix} x & 3 \\ 7 & 2y-1 \end{bmatrix} = \begin{bmatrix} 5-y & 3 \\ 7 & 3 \end{bmatrix}$

$x = 5-y$ and $2y-1 = 3$

$x = 5-2$ $\quad 2y = 4$

$\boxed{x = 3}$ $\quad \boxed{y = 2}$

Exp Find x, y, z if $3\begin{bmatrix} x & y \\ y & z \end{bmatrix} + 2\begin{bmatrix} 2x & -y \\ 3y & -4z \end{bmatrix} = \begin{bmatrix} 14 & 4-y \\ 18 & 15 \end{bmatrix}$

$$\begin{bmatrix} 3x & 3y \\ 3y & 3z \end{bmatrix} + \begin{bmatrix} 4x & -2y \\ 6y & -8z \end{bmatrix} = \begin{bmatrix} 14 & 4-y \\ 18 & 15 \end{bmatrix}$$

$3z + ^-8z = 15$

$-5z = 15$

$\boxed{z = -3}$

$3x + 4x = 14$

$7x = 14$

$\boxed{x = 2}$

$3y + ^-2y = 4-y$

$y = 4-y$

$2y = 4$

$\boxed{y = 2}$