

10.3 The Integral Test.

[6] $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$, use the integral test.

let $f(x) = \frac{1}{x(\ln x)^2}$, $f(x)$ is +ve, continuous & decreasing $\forall x \geq 2$

$\star f(x)$ is decreasing since:

$$f'(x) = \frac{-\left[2 \frac{x \ln x}{x} + (\ln x)^2\right]}{x^2(\ln x)^4} = \frac{-2 \ln x}{x^2(\ln x)^4} - \frac{(\ln x)^2}{x^2(\ln x)^4}$$
$$= -\frac{2}{x^2(\ln x)^3} - \frac{1}{x^2(\ln x)^2}$$

f' sign $\xrightarrow{2}$

Now, To find $\int_2^{\infty} \frac{1}{x(\ln x)^2} dx$

$$\Rightarrow \int_{\ln 2}^{\infty} \frac{1}{u^2} du = \lim_{b \rightarrow \infty} \int_{\ln 2}^b u^{-2} du$$

$$= \lim_{b \rightarrow \infty} \left(-\frac{1}{u} \Big|_{\ln 2}^b \right) = \lim_{b \rightarrow \infty} \left(-\frac{1}{b} + \frac{1}{\ln 2} \right)$$

$$= \frac{1}{\ln 2} \text{ Converges.}$$

$\therefore \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ Converges by integral test.

[13] $\sum_{n=1}^{\infty} \frac{n}{n+1}$

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n}{n+1} = 1 \neq 0.$$

by n -th term test $\sum_{n=1}^{\infty} \frac{n}{n+1}$ diverges.

let $u = \ln x \Rightarrow du = \frac{dx}{x}$

when $x=2 \Rightarrow u = \ln 2$
 $x=\infty \Rightarrow u = \infty$

20 $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$

Remember :-

$\ln x > 0 \quad x > 1$

Try to use the integral test.

Let $f(x) = \frac{\ln x}{\sqrt{x}}$, $f(x)$ is positive, continuous & decreasing $\forall x > 8$.

Since $f'(x) = \frac{\frac{\sqrt{x}}{x} - \frac{\ln x}{2\sqrt{x}}}{(\sqrt{x})^2} = \frac{2 - \ln x}{2x^{3/2}}$

$2 - \ln x < 0$

$\ln x > 2$

$x > e^2$

$f'_{\text{sign}} \frac{1}{2} + \frac{0}{e^2} \rightarrow$

$e^2 \approx 7.4$

$\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}} = \sum_{n=2}^7 \frac{\ln n}{\sqrt{n}} + \sum_{n=8}^{\infty} \frac{\ln n}{\sqrt{n}}$

So, we can apply the integral test for $n \geq 8$.

Now, $\int_8^{\infty} \frac{\ln x}{\sqrt{x}} dx$

$= \int_{\ln 8}^{\infty} \frac{u}{e^{u/2}} \cdot e^u du = \int_{\ln 8}^{\infty} u e^{u/2} du$

let $u = \ln x \Rightarrow du = \frac{dx}{x}$

$e^u = x \Rightarrow e^u du = dx$

$e^{u/2} = \sqrt{x}$

when $x=8 \Rightarrow u = \ln 8$

$x = \infty \Rightarrow u = \infty$

$= \lim_{b \rightarrow \infty} \int_{\ln 8}^b u e^{u/2} du$

$= \lim_{b \rightarrow \infty} \left(2u e^{u/2} - 4e^{u/2} \right) \Big|_{\ln 8}^b$

$= \lim_{b \rightarrow \infty} \left(2e^{u/2} (u-2) \right) \Big|_{\ln 8}^b$

$= \lim_{b \rightarrow \infty} \left(2e^{b/2} (b-2) - 2e^{\ln 8/2} (\ln 8 - 2) \right)$

$= \infty$ diverges.

by the integral test $\sum_{n=8}^{\infty} \frac{\ln n}{\sqrt{n}}$ diverges.

So, $\sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}}$ diverges.

deriv	Integral
u	$e^{u/2}$
1	$2e^{u/2}$
0	$4e^{u/2}$

22 $\sum_{n=1}^{\infty} \frac{5^n}{4^n + 3}$

Use n-th term test:

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{5^n}{4^n + 3} \quad \frac{\infty}{\infty}$$

$$= \lim_{n \rightarrow \infty} \frac{\ln 5 \cdot 5^n}{\ln 4 \cdot 4^n + 0} = \frac{\ln 5}{\ln 4} \lim_{n \rightarrow \infty} \left(\frac{5}{4}\right)^n$$

$$\frac{5}{4} > 1$$

$$= \infty \neq 0$$

$\Rightarrow$ by n-th term test the series diverges.

28 $\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^n$

by using n-th term test:

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e \neq 0$$

So the series diverges.

32 $\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln^2 n)}$

let $f(x) = \frac{1}{x(1 + \ln^2 x)}$ $\star$ [positive, continuous & decreasing for $x > 1$]

$$f'(x) = \frac{-(x \cdot 2 \ln x \cdot \frac{1}{x} + (1 + \ln^2 x))}{x^2(1 + \ln^2 x)^2} = \frac{-2 \ln x - (1 + \ln^2 x)}{x^2(1 + \ln^2 x)^2}$$

$$= -\frac{2 \ln x}{x^2(1 + \ln^2 x)^2} - \frac{1}{x^2(1 + \ln^2 x)^2}$$

Now, $\int \frac{1}{x(1 + \ln^2 x)} dx$

$$= \int_0^{\infty} \frac{du}{1 + u^2} = \lim_{b \rightarrow \infty} \int_0^b \frac{du}{1 + u^2}$$

$$= \lim_{b \rightarrow \infty} \left(\tan^{-1} u \Big|_0^b \right) = \lim_{b \rightarrow \infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2} \quad \text{Converges.}$$

let $u = \ln x \Rightarrow du = \frac{dx}{x}$

when $x = 1 \Rightarrow u = 0$

$x = \infty \Rightarrow u = \infty$

So, by the integral test $\sum_{n=1}^{\infty} \frac{1}{n(1 + \ln^2 n)}$ converges.

38 $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

let $f(x) = \frac{x}{x^2+1}$

$$f'(x) = \frac{(x^2+1) - 2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$$

f' sign $\begin{array}{c} \text{---} \text{---} \text{---} \\ \text{---} \end{array}$

So, $f(x)$ is continuous, positive & decreasing $\forall x \geq 1$.

$$\begin{aligned} \int_1^{\infty} \frac{x}{x^2+1} &= \lim_{b \rightarrow \infty} \int_1^b \frac{x}{x^2+1} \\ &= \lim_{b \rightarrow \infty} \left(\frac{1}{2} \ln|x^2+1| \Big|_1^b \right) = \lim_{b \rightarrow \infty} \left(\frac{1}{2} \ln(b^2+1) - \frac{1}{2} \ln 2 \right) \\ &= \infty \text{ diverges.} \end{aligned}$$

So, by integral test the series diverges.

40 $\sum_{n=1}^{\infty} \text{sech}^2 n$

let $f(x) = \text{sech}^2 x = \frac{1}{\cosh^2 x} = \left(\frac{2}{e^x + e^{-x}} \right)^2 = \frac{4}{(e^x + e^{-x})^2}$

$$f'(x) = -\frac{8(e^x - e^{-x})}{(e^x + e^{-x})^3}$$

So, $f(x)$ is positive, continuous & decreasing $\forall x \geq 1$.

$$\begin{aligned} \text{Now, } \int_1^{\infty} \text{sech}^2 x \, dx &= \lim_{b \rightarrow \infty} \int_1^b \text{sech}^2 x \, dx \\ &= \lim_{b \rightarrow \infty} \left(\tanh x \Big|_1^b \right) = \lim_{b \rightarrow \infty} (\tanh b - \tanh 1) \\ &= 1 - \tanh 1 \\ &\text{converges.} \end{aligned}$$

So, by the integral test

$\sum_{n=1}^{\infty} \text{sech}^2 n$ converges.

Remember:-

$$\tanh b = \frac{e^b - e^{-b}}{e^b + e^{-b}}$$

$$\lim_{b \rightarrow \infty} \frac{1 - \frac{e^{-b}}{e^b}}{1 + \frac{e^{-b}}{e^b}} = \frac{1-0}{1+0} = 1$$

42 For what values of a do the series converge??

$$\sum_{n=3}^{\infty} \left(\frac{1}{n-1} - \frac{2a}{n+1} \right)$$

by using the integral test:

$$\begin{aligned} \int_3^{\infty} \left(\frac{1}{x-1} - \frac{2a}{x+1} \right) dx &= \lim_{b \rightarrow \infty} \int_3^b \left(\frac{1}{x-1} - \frac{2a}{x+1} \right) dx \\ &= \lim_{b \rightarrow \infty} \left(\ln|x-1| - 2a \ln|x+1| \right) \Big|_3^b \\ &= \lim_{b \rightarrow \infty} \left(\ln \left| \frac{x-1}{(x+1)^{2a}} \right| \right) \Big|_3^b \\ &= \lim_{b \rightarrow \infty} \left(\ln \left| \frac{b-1}{(b+1)^{2a}} \right| - \ln \left| \frac{2}{4^{2a}} \right| \right) \end{aligned}$$

$$\lim_{b \rightarrow \infty} \ln \left(\frac{b-1}{(b+1)^{2a}} \right) = \lim_{b \rightarrow \infty} \ln \frac{1}{2a(b+1)^{2a-1}}$$

$$= \begin{cases} 0 & , a = \frac{1}{2} \\ \infty & , a < \frac{1}{2} \end{cases}$$

If $a > \frac{1}{2}$, the terms of the series become negative & the integral test does not apply.

but when $a > \frac{1}{2}$, the series behaves like a negative multiple of the harmonic series, & so, it's diverges.

$$\text{So, } \sum_{n=3}^{\infty} \frac{1}{n-1} - \frac{2a}{n+1} \text{ converges only when } \underline{a = \frac{1}{2}}$$

10.4 $\Rightarrow$ Comparison Tests

8 Use D.C.T to determine if the following sequence converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$$

Compare with $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Note that $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$ & $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ have nonnegative terms.

$$\Rightarrow \text{for } n \geq 1 \Rightarrow \sqrt{n} \geq 1$$

$$2\sqrt{n} \geq 2$$

$$(2\sqrt{n}+1 \geq 3) \quad *n$$

$$2n\sqrt{n}+n \geq 3n \geq 3$$

add n^2 for both sides.

$$2n\sqrt{n}+n+n^2 \geq n^2+3$$

$$n(n+2\sqrt{n}+1) \geq n^2+3$$

$$n(\sqrt{n}+1)^2 \geq n^2+3$$

$$\frac{n(\sqrt{n}+1)^2}{n^2+3} \geq 1$$

take the square root for the both sides

$$\frac{\sqrt{n}(\sqrt{n}+1)}{\sqrt{n^2+3}} \geq 1$$

$$\frac{\sqrt{n}+1}{\sqrt{n^2+3}} \geq \frac{1}{\sqrt{n}}$$

(both series have nonnegative terms for $n \geq 1$).

but, we know that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series, since $p = \frac{1}{2} \leq 1$

So, by D.C.T $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{\sqrt{n^2+3}}$ diverges.

15 Use L.C.T to determine if the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

★ Compare with $\sum_{n=2}^{\infty} \frac{1}{n}$, which is divergent p-series, since $p=1 \leq 1$.

★ $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ & $\sum_{n=2}^{\infty} \frac{1}{n}$ have positive terms for $n \geq 2$.

★ Now, $\lim_{n \rightarrow \infty} \frac{\frac{1}{\ln n}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{\ln n} = \lim_{n \rightarrow \infty} \frac{1}{1/n} = \infty$.

$\Rightarrow$ by L.C.T $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ diverges.

18 Use any method to determine if the following series converges or diverges

$$\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$$

★ Let Use L.C.T & compare with $\sum_{n=1}^{\infty} \frac{1}{n}$

★ Both series have positive terms.

★ $\sum_{n=1}^{\infty} \frac{1}{n}$ divergent (harmonic series).

★ $\lim_{n \rightarrow \infty} \frac{\frac{3}{n+\sqrt{n}}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{3n}{n+\sqrt{n}} = 3 > 0$.

So, both series converge or both diverge.

but, we say that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent

So, $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ is also divergent.

by
L.C.T

By D.C.T

$$n+n+n > n+\sqrt{n}+0$$

$$3n > n+\sqrt{n}$$

$$\frac{3}{n+\sqrt{n}} > \left(\frac{1}{n}\right)$$

$\sum \frac{1}{n}$ dive
So by D.C.T $\sum \frac{3}{n+\sqrt{n}}$ div.

27 $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$

Compare with $\sum_{n=3}^{\infty} \frac{1}{n}$

Note that: $\sum_{n=3}^{\infty} \frac{1}{n}$ & $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$ have positive terms.

$\Rightarrow n > \ln n$ take $\ln$ for both sides.

$$\ln n > \ln(\ln n)$$

$$\& \underline{n} > \ln n > \underline{\ln(\ln n)}.$$

$$\text{So, } \frac{1}{n} < \frac{1}{\ln(\ln n)}$$

$$\sum_{n=3}^{\infty} \frac{1}{n} < \sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$$

but, we know that $\sum_{n=3}^{\infty} \frac{1}{n}$ diverges (harmonic series).

So, by D.C.T $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$ diverges.

28 $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$

Let use L.C.T & compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Note: $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ & $\sum_{n=1}^{\infty} \frac{1}{n^2}$ have nonnegative terms

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series, $p > 1$)

$$\underline{\text{Now,}} \quad \lim_{n \rightarrow \infty} \frac{\frac{(\ln n)^2}{n^3}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{(\ln n)^2}{n^3} \cdot n^2 = \lim_{n \rightarrow \infty} \frac{(\ln n)^2}{n} \quad \frac{\infty}{\infty}$$

$$= \lim_{n \rightarrow \infty} \frac{2 \ln n \cdot \left(\frac{1}{n}\right)}{1} = \lim_{n \rightarrow \infty} \frac{2 \ln n}{n} \quad \frac{\infty}{\infty}$$

$$= \lim_{n \rightarrow \infty} \frac{2 \left(\frac{1}{n}\right)}{1} = \lim_{n \rightarrow \infty} \frac{2}{n} = 0.$$

So, by L.C.T $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ converges.

32 $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$

Use L.C.T & compare with $\sum_{n=2}^{\infty} \frac{1}{n+1}$

$$\lim_{n \rightarrow \infty} \frac{\ln(n+1)}{\frac{1}{n+1}} = \lim_{n \rightarrow \infty} \ln(n+1) = \infty$$

★ $\sum_{n=2}^{\infty} \frac{1}{n+1}$ (use the integral test) $\Rightarrow f(x) = \frac{1}{x+1}$ (true, cont. & decreasing)

↳ have positive terms.

$$\text{now, } \int_2^{\infty} \frac{1}{x+1} dx = \lim_{b \rightarrow \infty} \int_2^b \frac{dx}{x+1} = \lim_{b \rightarrow \infty} \left(\ln|x+1| \right) \Big|_2^b$$

$$= \lim_{b \rightarrow \infty} (\ln|b+1| - \ln 3)$$

$= \infty$ diverges.

So, $\sum_{n=2}^{\infty} \frac{1}{n+1}$ diverges by integral test.

So, by L.C.T $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$ diverges.

Note: You can use the integral test to show that

$\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1}$ diverges.

explain: $\int_2^{\infty} \frac{\ln(x+1)}{x+1} dx = \int_{\ln 3}^{\infty} u du$

$$= \lim_{b \rightarrow \infty} \left[\frac{1}{2} u^2 \right]_{\ln 3}^b = \lim_{b \rightarrow \infty} \frac{1}{2} (b^2 - \ln^2 3) = \infty$$

40 $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n}$

Let Compare with $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$ & use D.C.T.

* $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n}$ & $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$ have nonnegative terms.

Now, $\frac{2^n + 3^n}{3^n + 4^n} < \frac{2^n + 3^n}{4^n}$

$$\sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n} < \sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n} = \sum_{n=1}^{\infty} \left[\left(\frac{2}{4}\right)^n + \left(\frac{3}{4}\right)^n \right]$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$$

(Geometric serieses with $|r| < 1$)

$$= \frac{\frac{1}{2}}{1 - \frac{1}{2}} + \frac{\frac{3}{4}}{1 - \frac{3}{4}}$$

$$= \frac{\frac{1}{2}}{\frac{1}{2}} + \frac{\frac{3}{4}}{\frac{1}{4}} = 1 + 3 = \underline{\underline{4}}$$

So, $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{4^n}$ Converges.

By D.C.T $\sum_{n=1}^{\infty} \frac{2^n + 3^n}{3^n + 4^n}$ Converges.

Note :- We can use L.C.T

$$\lim_{n \rightarrow \infty} \frac{\frac{2^n + 3^n}{3^n + 4^n}}{\frac{2^n + 3^n}{4^n}} = \lim_{n \rightarrow \infty} \frac{4^n}{3^n + 4^n} = 1 > 0 \quad (\text{both converges})$$

43 $\sum_{n=2}^{\infty} \frac{1}{n!}$

Compare with

$\sum_{n=2}^{\infty} \frac{1}{n^2-n}$ & use D.C.T.

$$\sum_{n=2}^{\infty} \frac{1}{n!} < \sum_{n=2}^{\infty} \frac{1}{n^2-n}$$

$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^2-n}$ Converges.

So, $\sum_{n=2}^{\infty} \frac{1}{n!}$ Converges.

$$n! = n(n-1)(n-2) \dots 2 \cdot 1$$

$$n! > n(n-1)$$

$$\frac{1}{n!} < \frac{1}{n(n-1)} = \frac{1}{n^2-n}$$

$$\sum_{n=2}^{\infty} \frac{1}{n!} < \sum_{n=2}^{\infty} \frac{1}{n^2-n}$$

★ Now: Determine $\sum_{n=2}^{\infty} \frac{1}{n^2-n}$

Converges or diverges.

$\rightarrow$ Compare with $\sum_{n=2}^{\infty} \frac{1}{n^2}$.

$\left(\sum_{n=2}^{\infty} \frac{1}{n^2-n} \right)$ & $\sum_{n=2}^{\infty} \frac{1}{n^2}$ have nonnegative terms.

& $\sum_{n=2}^{\infty} \frac{1}{n^2}$ Converges (p-series, $p > 1$)

by L.C.T

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{n^2-n}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{n^2}{n^2-n} = 1 \neq 0$$

So, both Converges.

$\sum_{n=2}^{\infty} \frac{1}{n^2-n}$ Converges.

Note: You can use the ratio test.

52 $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$

Compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\left(\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ converges, p-series with } p=2 \right)$.

$\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$ & $\sum_{n=1}^{\infty} \frac{1}{n^2}$ have nonnegative terms.

$$\lim_{n \rightarrow \infty} \frac{\frac{\sqrt[n]{n}}{n^2}}{\frac{1}{n^2}} = \lim_{n \rightarrow \infty} \frac{\sqrt[n]{n}}{n^2} \cdot n^2$$

$$= \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1 > 0$$

So, by L.C.T $\sum_{n=1}^{\infty} \frac{\sqrt[n]{n}}{n^2}$ converges.