# Lead Compensation

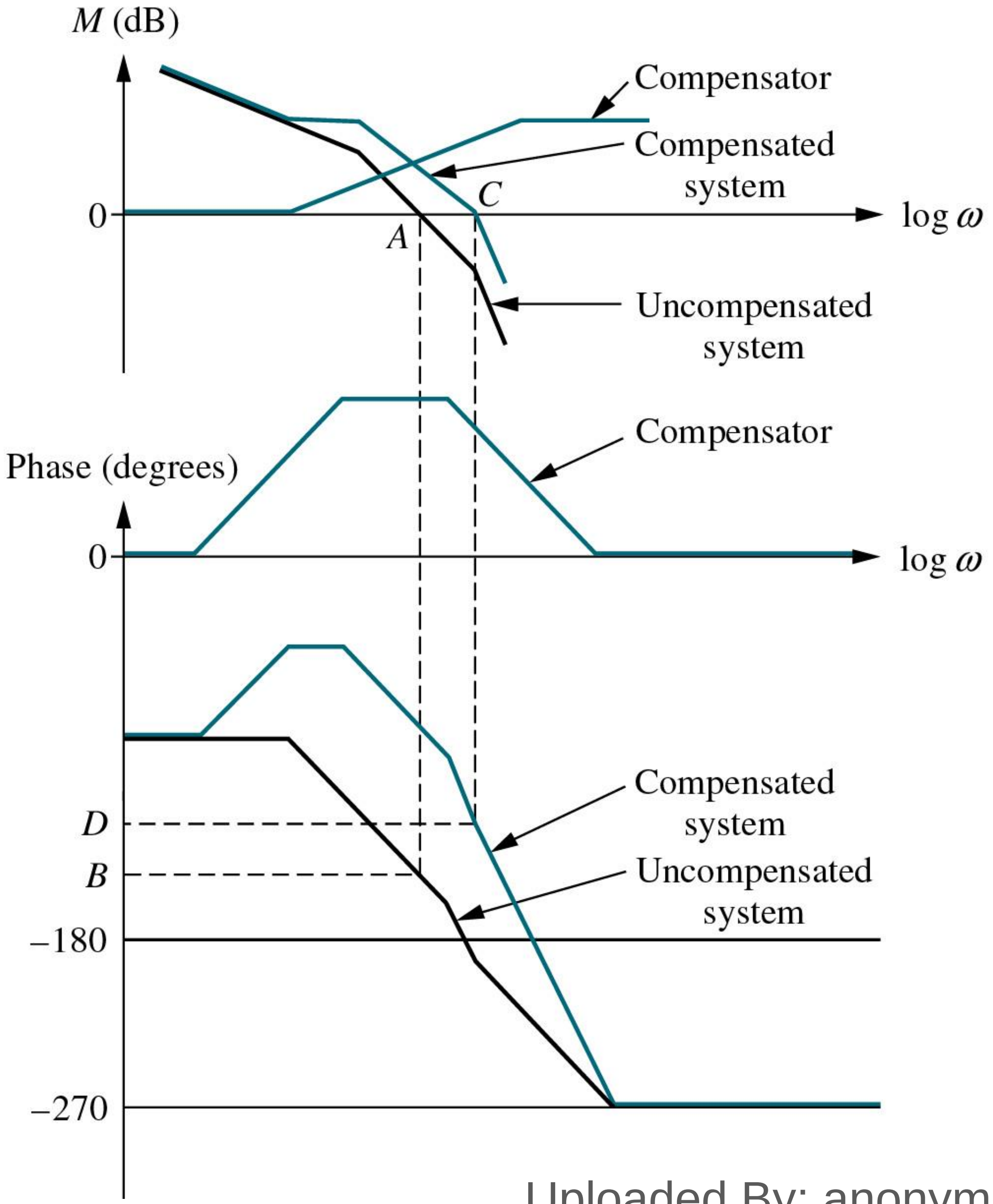
- The transfer function for the lead compensator is given by

$$C_c(s) = \frac{1}{\beta}\frac{(s+\frac{1}{T})}{(s+\frac{1}{\beta T})}, \qquad \text{where } \beta < 1$$

- The Dc gain = $\lim_{s \to 0} C_c(s) = \frac{1}{\beta}$
- The lead compensator increases the bandwidth by increasing the gain crossover frequency. At the same time, the phase diagram is raised at higher frequencies. The result is a larger phase margin and a higher phase-margin frequency.
- In the time domain, lower percent overshoots (larger phase margins) with smaller peak times (higher phase margin frequencies) are the results.
- Notice that the initial slope, which determines the steady-state error, is not affected by the design for the transient response.

- The uncompensated system has a small phase margin *(B)* and a low phase margin frequency *(A)*.

- Using a phase lead compensator, the phase angle plot (compensated system) is raised for higher frequencies. At the same time, the gain crossover frequency in the magnitude plot is increased from (A) rad/s to (C) rad/s. These effects yield a larger phase margin (D), a higher phase margin frequency (C), and a larger bandwidth.



- $C_c(jw) = \frac{1}{\beta}\frac{(jw+\frac{1}{T})}{(jw+\frac{1}{\beta T})}$ (11.6)

From Eq. (11.6) the phase angle of the lead compensator, $\phi_c$, is

$$\phi_c = \tan^{-1}\omega T - \tan^{-1}\omega\beta T \tag{11.7}$$

Differentiating with respect to $\omega$, we obtain

$$\frac{d\phi_c}{d\omega} = \frac{T}{1+(\omega T)^2} - \frac{\beta T}{1+(\omega\beta T)^2} \tag{11.8}$$

Setting Eq. (11.8) equal to zero, we find that the frequency, $\omega_{\text{max}}$, at which the maximum phase angle, $\phi_{\text{max}}$, occurs is

$$\boxed{\omega_{\text{max}} = \frac{1}{T\sqrt{\beta}}} \tag{11.9}$$

Substituting Eq. (11.9) into Eq. (11.6) with $s = j\omega_{max}$,

$$G_c(j\omega_{max}) = \frac{1}{\beta}\frac{j\omega_{max} + \frac{1}{T}}{j\omega_{max} + \frac{1}{\beta T}} = \frac{j\frac{1}{\sqrt{\beta}} + 1}{j\sqrt{\beta} + 1} \tag{11.10}$$

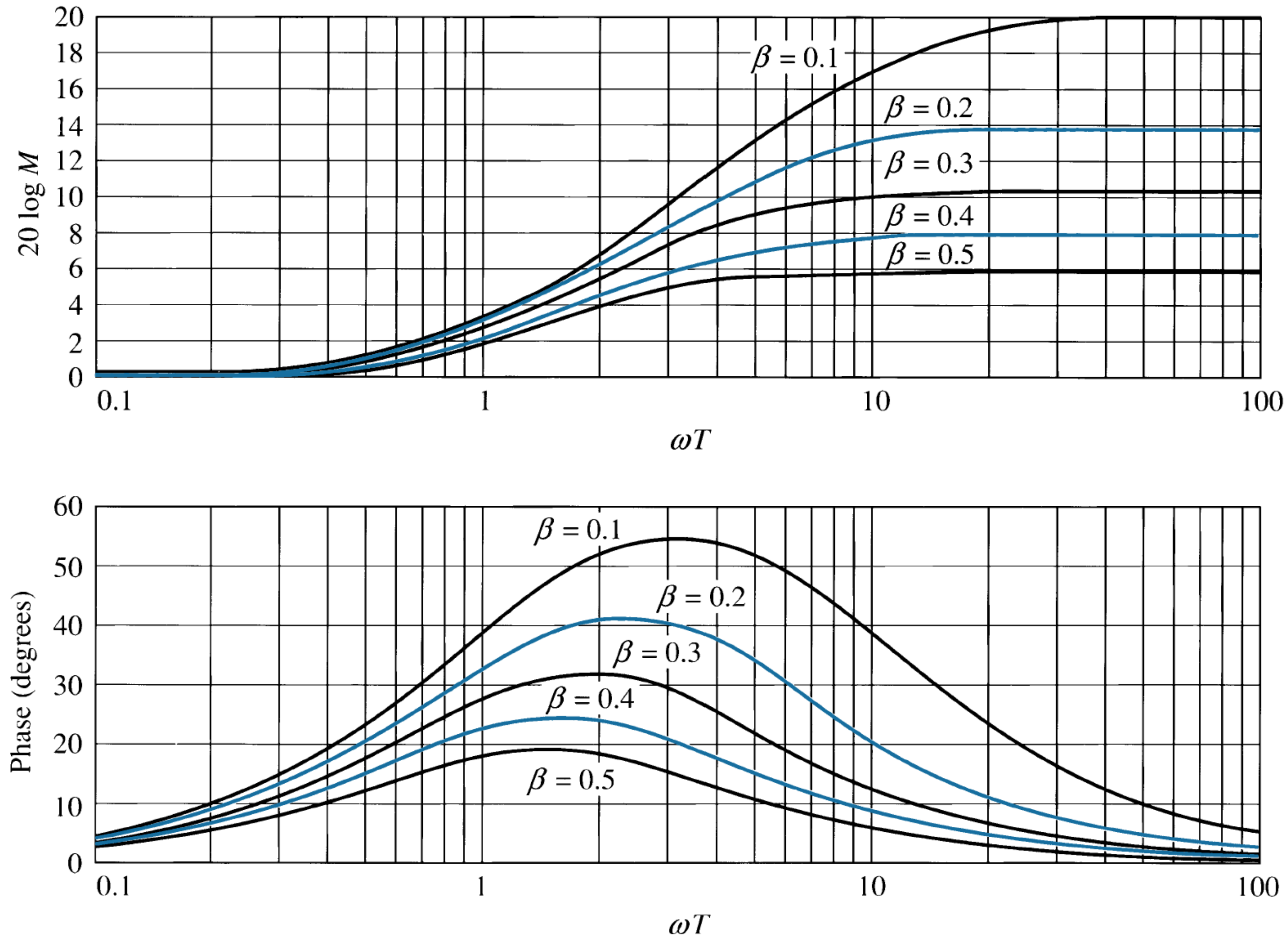
Making use of $\tan(\phi_1 - \phi_2) = (\tan\phi_1 - \tan\phi_2)/(1 + \tan\phi_1 \tan\phi_2)$, the maximum phase shift of the compensator, $\phi_{max}$, is

$$\boxed{\phi_{max} = \tan^{-1}\frac{1-\beta}{2\sqrt{\beta}} = \sin^{-1}\frac{1-\beta}{1+\beta}} \tag{11.11}$$

and the compensator's magnitude at $\omega_{max}$ is

$$\boxed{|G_c(j\omega_{max})| = \frac{1}{\sqrt{\beta}}} \tag{11.12}$$

Frequency response of a lead compensator




$$C_c(s) = \frac{1}{\beta}\frac{(s+\frac{1}{T})}{(s+\frac{1}{\beta T})}, \text{where } \beta < 1$$

## Design Procedure

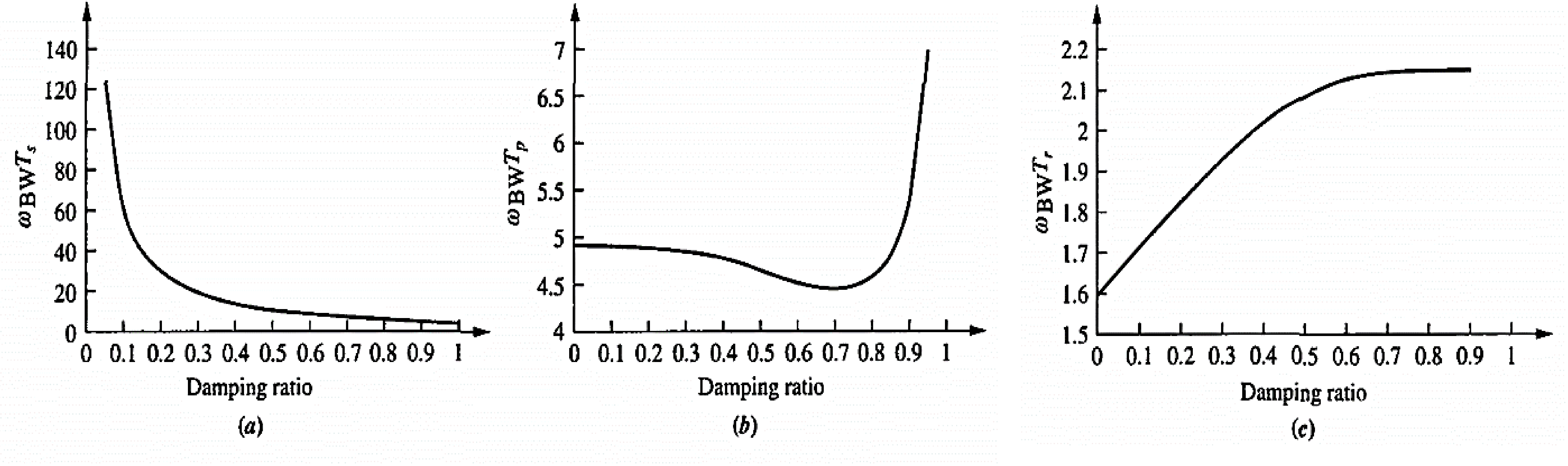
1. Find the closed-loop bandwidth required to meet the settling time, peak time, or rise time requirement (see Eq’s (10.54) through (10.56)).

$$\omega_{BW} = \omega_n \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \qquad (10.54)$$

$$\omega_{BW} = \frac{4}{T_s \zeta} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \qquad (10.55)$$

$$\omega_{BW} = \frac{\pi}{T_p \sqrt{1-\zeta^2}} \sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} \qquad (10.56)$$

You can use the following figures instead of Eq’s (10.54-10.56)




(a) (b) (c)

2. Since the lead compensator has negligible effect at low frequencies, set the gain, *K*, of the uncompensated system to the value that satisfies the steady state error requirement.

3. Plot the Bode magnitude and phase diagrams for this value of gain and determine the uncompensated system's phase margin.

4. Find the phase margin to meet the damping ratio or percent overshoot requirement. Then evaluate the additional phase contribution required from the compensator.

5. Determine the value of $p$ (see Eq's. (11.6) and (11.11)) from the lead compensator's required phase contribution.

6. Determine the compensator's magnitude at the peak of the phase curve (Eq. (11.12)).

7. Determine the new phase-margin frequency by finding where the uncompensated system's magnitude curve is the negative of the lead compensator's magnitude at the peak of the compensator's phase curve.

8. Design the lead compensator's break frequencies, using Eq’s. (11.6) and (11.9) to find T and the break frequencies.

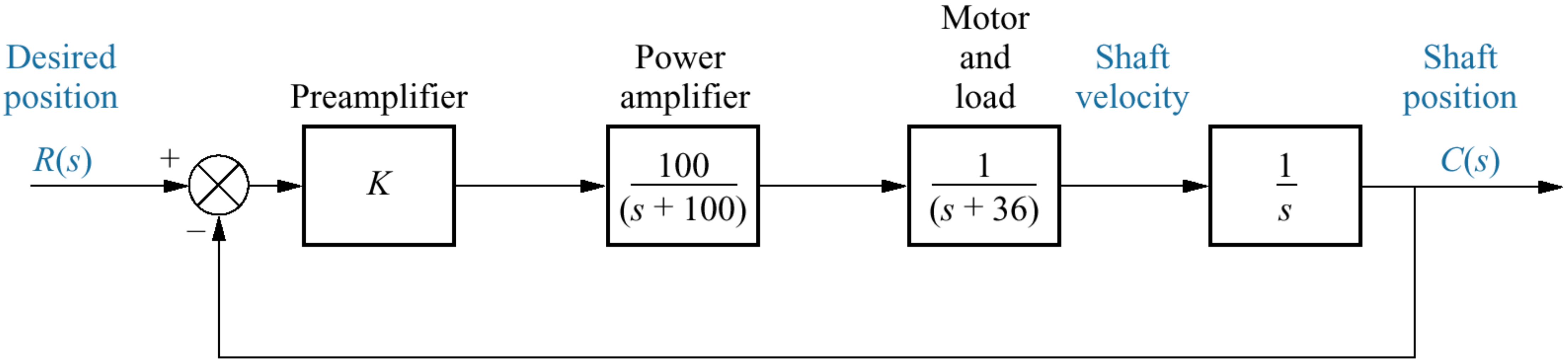
9. Reset the system gain to compensate for the lead compensator's gain.

10. Check the bandwidth to be sure the speed requirement in Step 1 has been met.

11. Simulate to be sure all requirements are met.

12. Redesign if necessary to meet requirements.

Problem: Given the system of Figure 11.2, design a lead compensator to yield a 20% overshoot and $K_v = 40$, with a peak time of 0.1 second.



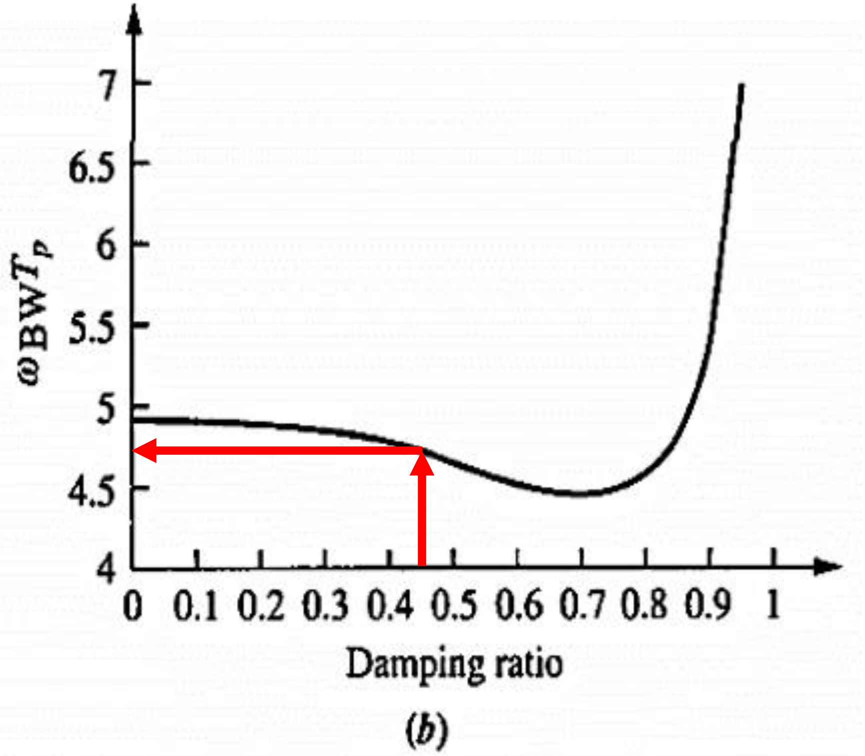

The uncompensated system is $G(s) = \frac{100\,K}{s(s+36)(s+100)}$

1. Find the closed loop bandwidth ($\omega_{BW}$)

$T_p = 0.1$ second      OS%=20% ⟶ $\zeta$=0.456

$$\omega_{BW} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

From the following figure:




$\omega_{BW}T_p = 4.66$

$\omega_{BW} = \frac{4.66}{0.1} = 46.6$ rad/s

## 2. Find the pre-gain K:

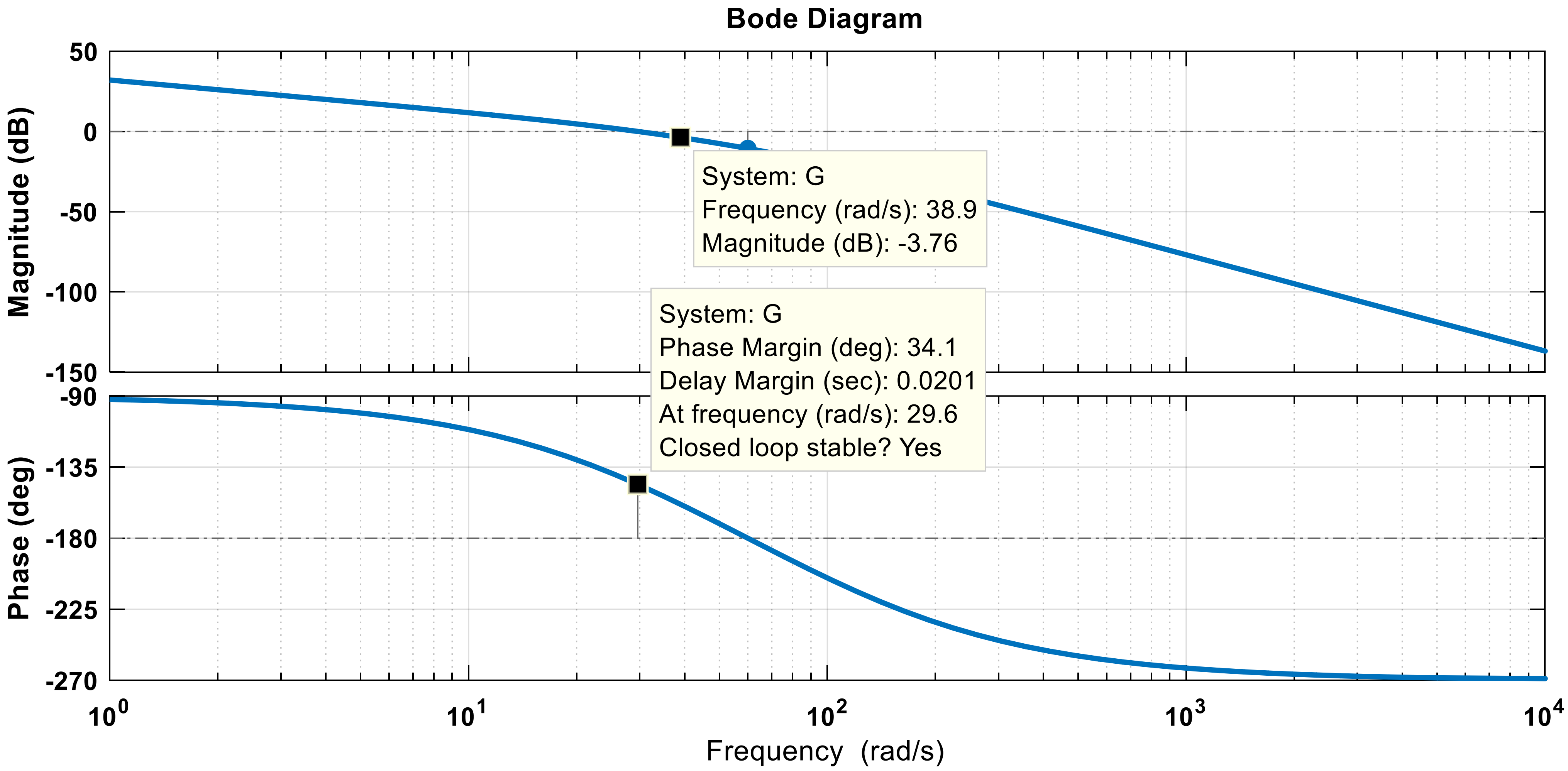
In order to meet the specification of $K_v = 40$

$$K_v = \lim_{s \to 0} s\,G(s) = \frac{\cancel{s}\,100\,K}{\cancel{s}(s+36)(s+100)} =$$

$$K_v = \quad 40 = \quad \frac{100\,K}{(36)(100)}$$

$K$=1440



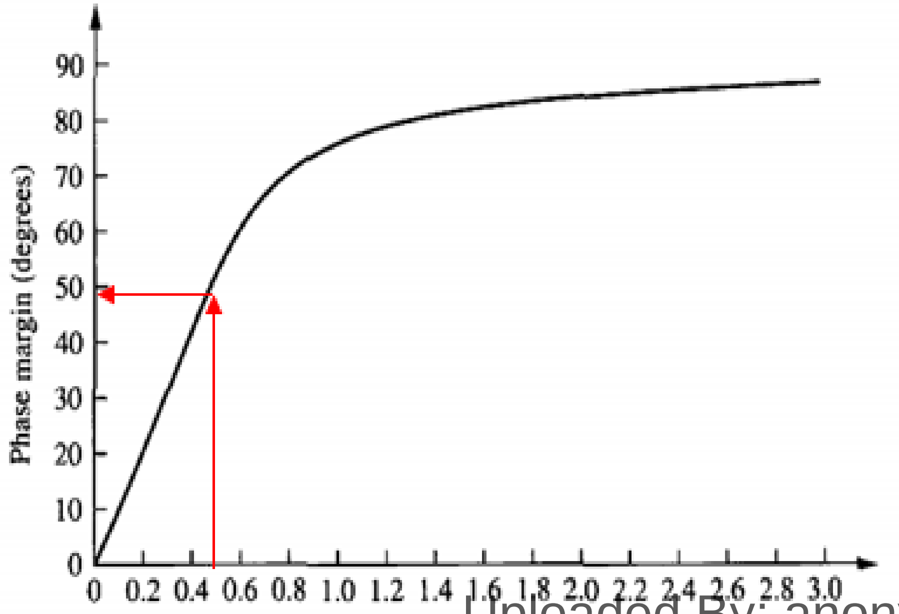

3. Plot the bode plot for the uncompensated system for $K = 1440$ are

$$G(s) = \frac{144{,}000}{s(s+36)(s+100)}$$

## 4. Compute the Lead compensator phase margin $\phi_{max}$

- A 20% overshoot gives $\zeta$=0.456. Also the required phase margin PM=48.1°.
- For the uncompensated system with $K$ = 1440 the phase margin is 34° at a phase-margin frequency $\omega_{PM}$=29.6 rad/s. See the bode plots
- We need to add 5-12 to the phase margin to compensate the phase angle for the lead compensator.
- PM = PM+10 = 48.1+10 = 58.1
- The total phase contribution required from the compensator is:
  $\phi_{max}$=58.1-34 = 24.1

- If the system's characteristics are not acceptable after the design, then a redesign with a different correction factor may be necessary.



5. Determine the value of $\beta$

$$\phi_{max} = \tan^{-1}\frac{1-\beta}{2\sqrt{\beta}} = \sin^{-1}\frac{1-\beta}{1+\beta} \longrightarrow \beta = 0.42 \quad for\ \phi_{max} = 24.1$$

6. Determine the compensator's magnitude at $\omega_{max}$

$$|G_c(j\omega_{max})| = \frac{1}{\sqrt{\beta}} = 3.76 \text{ dB}$$

7. Determine the new phase-margin frequency $\omega_{max}$.

- If we select $\omega_{max}$ to be the new phase-margin frequency, the uncompensated system's magnitude at this frequency must be -3.76 dB to yield a 0 dB crossover at $\omega_{max}$ for the compensated system.

- The uncompensated system passes through -3.76 dB at $\omega_{max}$ = 39 rad/s. This frequency is thus the new phase-margin frequency.

8. Design the lead compensator's break frequencies (T) fom (Eq. 11.9)

$$\omega_{max} = \frac{1}{T\sqrt{\beta}} \longrightarrow \text{for } \beta=0.42 \text{ and } \omega_{max}=39 \text{ rad/sec} \longrightarrow \frac{1}{T} = 25.3 \ , \frac{1}{\beta T} = 60.2$$

9. Hence, the compensator is given by:

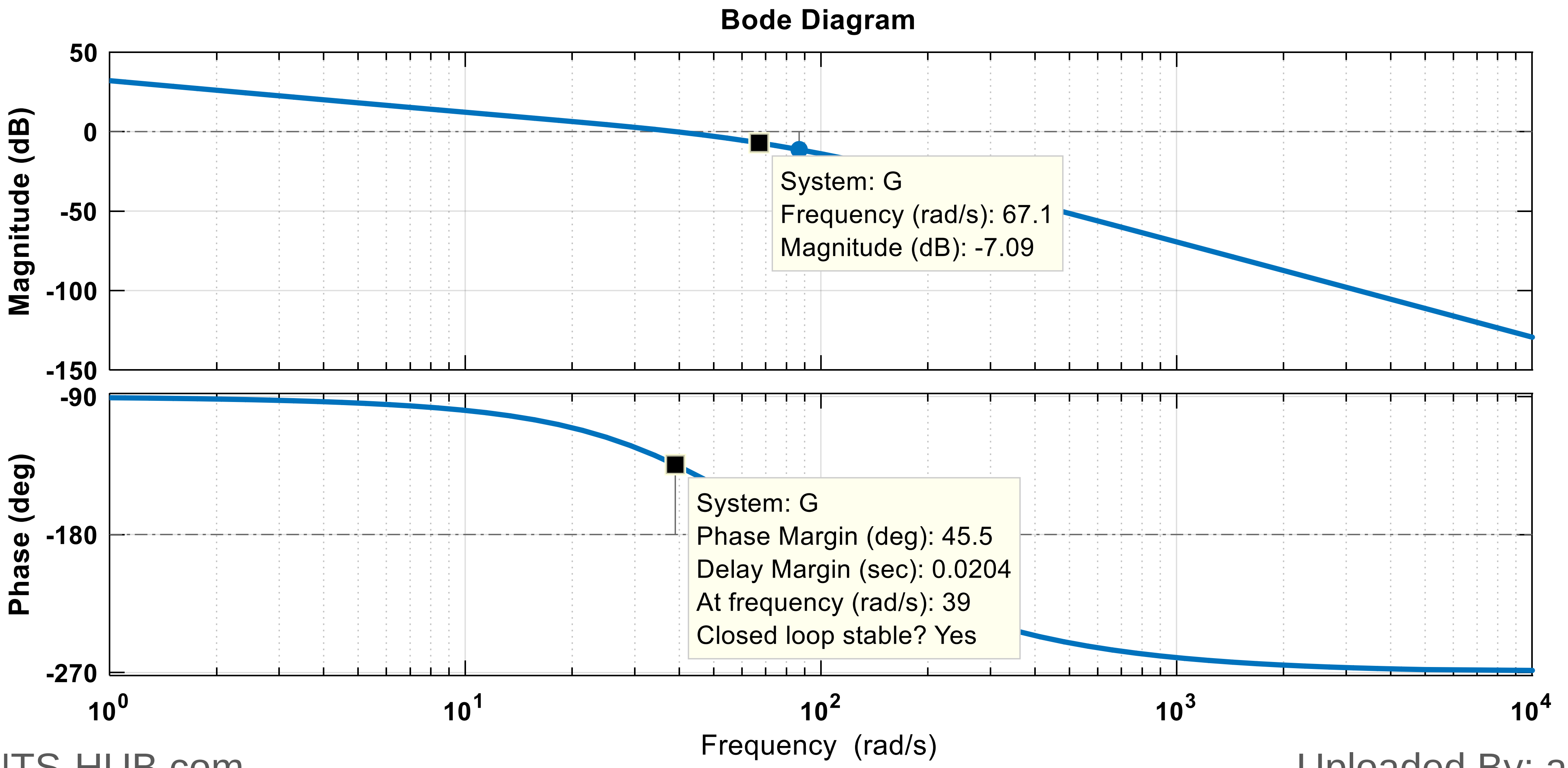
$$C_c(s) = \frac{1}{\beta}\frac{(s+\frac{1}{T})}{(s+\frac{1}{\beta T})} = 2.38\frac{(s+25.3)}{(s+60.2)}$$

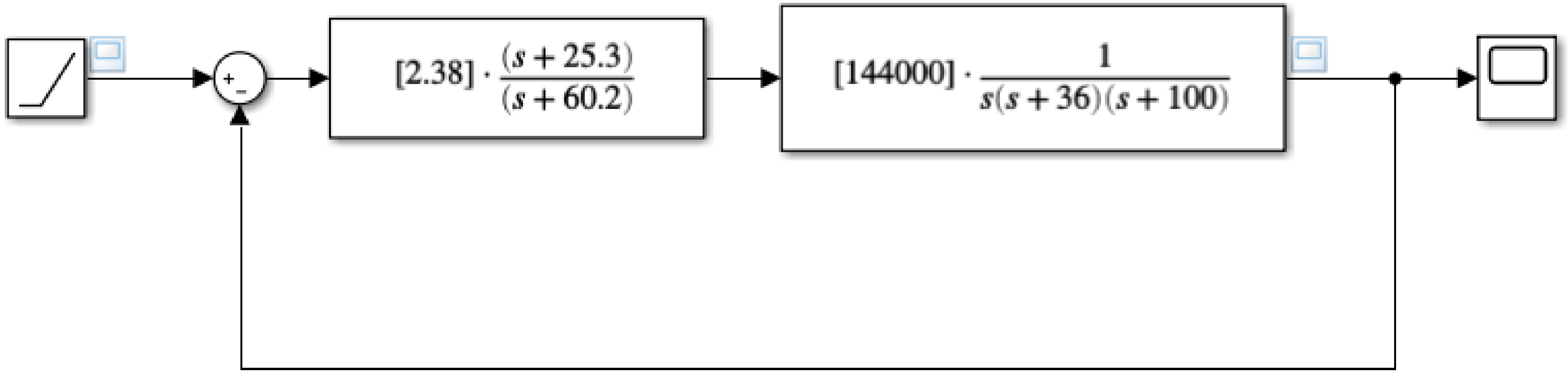
10. Check the bandwidth for the system

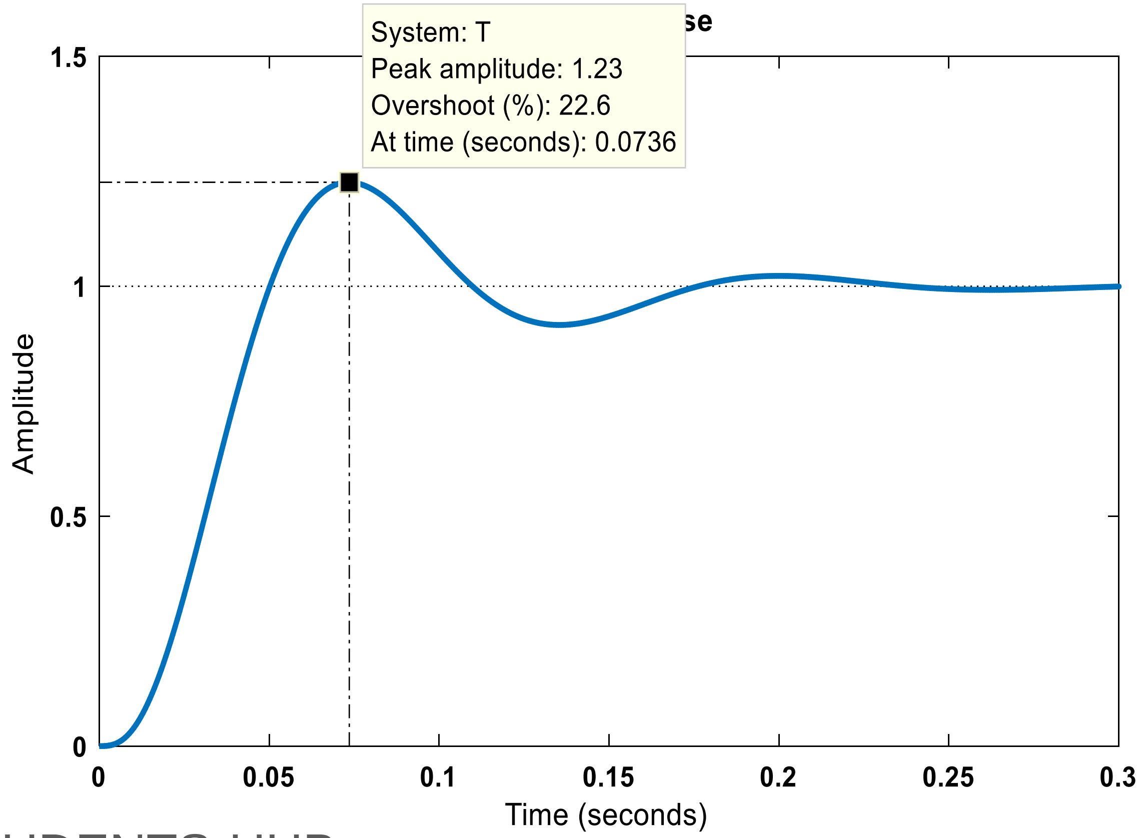
$$G_{all}(\mathrm{s})=C_c(s)G(s)=2.38\frac{(s+25.3)}{(s+60.2)} * \frac{144{,}000}{s(s+36)(s+100)}=\frac{342600(s+25.3)}{s(s+36)(s+100)(s+60.2)}$$

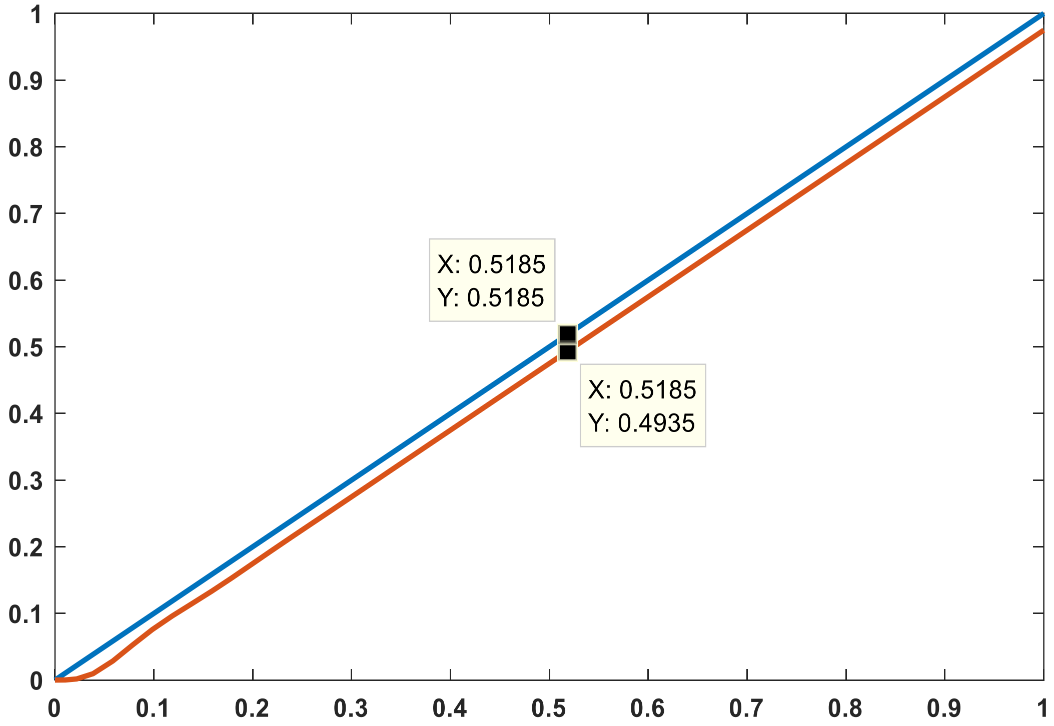
# 10. Check the bandwidth frequency $\omega_{BW}$

$$G_{all}(s) = \frac{342600(s+25.3)}{s(s+36)(s+100)(s+60.2)}$$

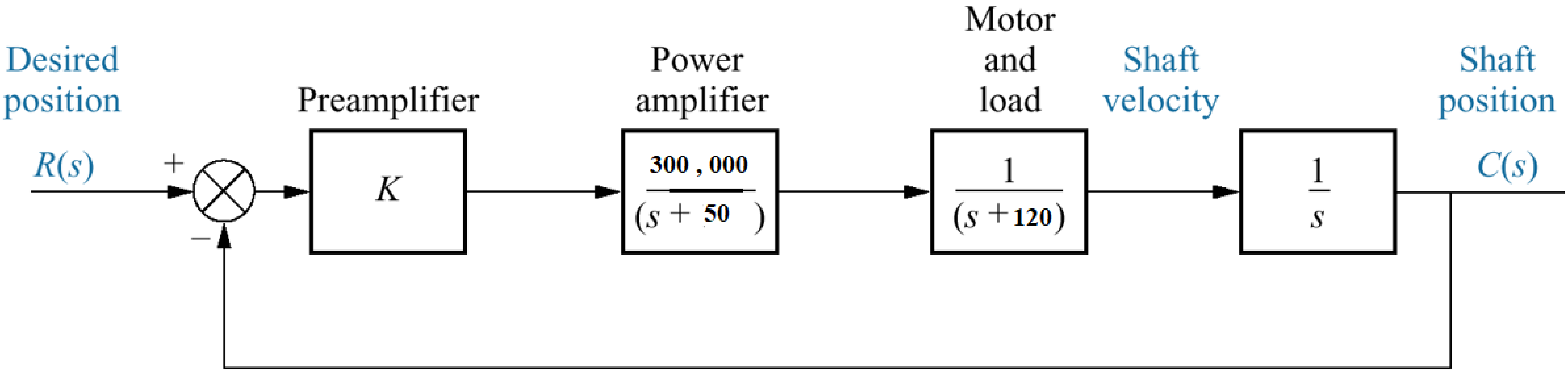



[2.38] · (s + 25.3)/(s + 60.2)
[144000] · 1/(s(s + 36)(s + 100))


System: T
Peak amplitude: 1.23
Overshoot (%): 22.6
At time (seconds): 0.0736
Amplitude
Time (seconds)
1.5
1
0.5
0
0.05
0.1
0.15
0.2
0.25
0.3


X: 0.5185
Y: 0.5185
X: 0.5185
Y: 0.4935

Problem Design a lead compensator for the system to meet the following specifications: $\%OS$ = 20%, $Ts$ =0.2 s and $Kv$ = 100.



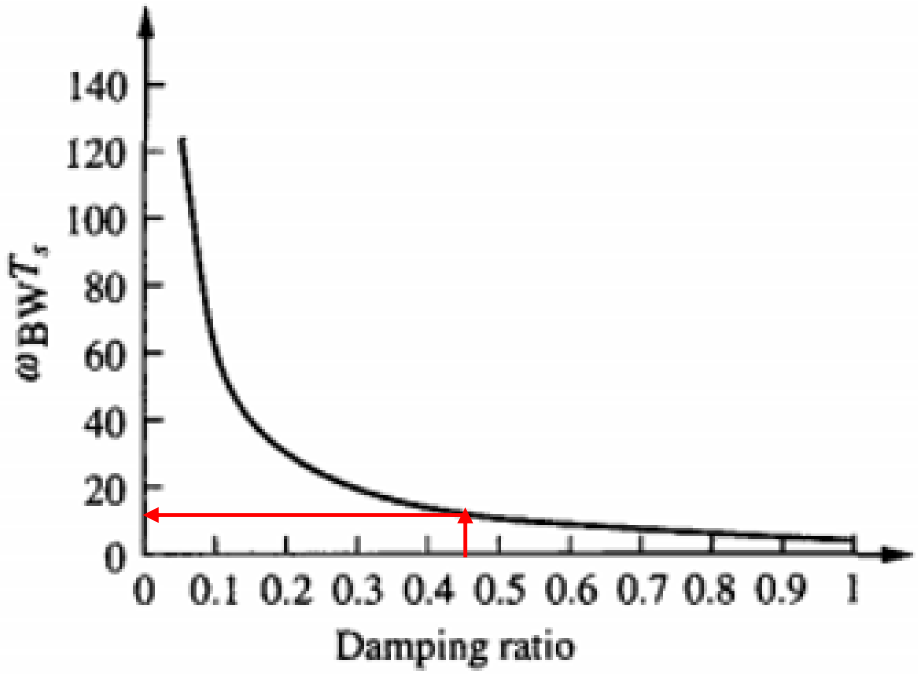

$$G_{un}(s) = \frac{300{,}000}{s(s+50)(s+120)}$$

1. Find the closed loop bandwidth ($\omega_{BW}$)

$T_s = 0.2$ second      OS%=20%  ⟶  $\zeta$=0.456

$$\omega_{BW} = \frac{4}{T_s\zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

From the following figure:




$\omega_{BW}T_s$=15

$\omega_{BW}$=$\frac{15}{0.2}$ = 75 rad/s

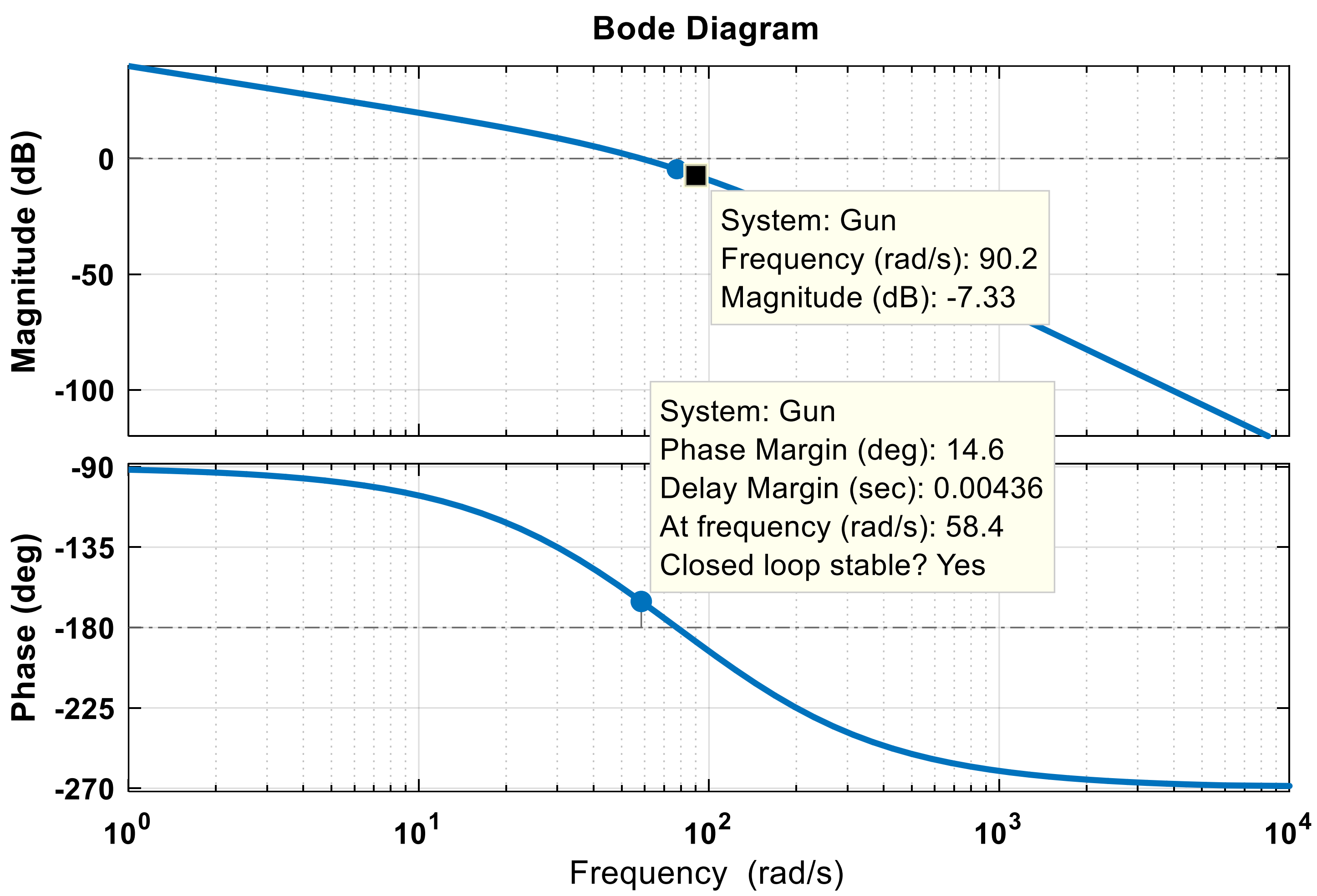
## 2. Find the pre-gain K:

In order to meet the specification of $K_v = 100$

$$K_v = \lim_{s \to 0} s\, G(s) = \lim_{s \to 0} \frac{s\ 30000\ K}{s(s+50)(s+120)}$$

$$K_v = \quad 100 = \quad \frac{300000\ K}{(50)(120)}$$

K=2




3. Plot the bode plot for the uncompensated system for $K = 2$ are

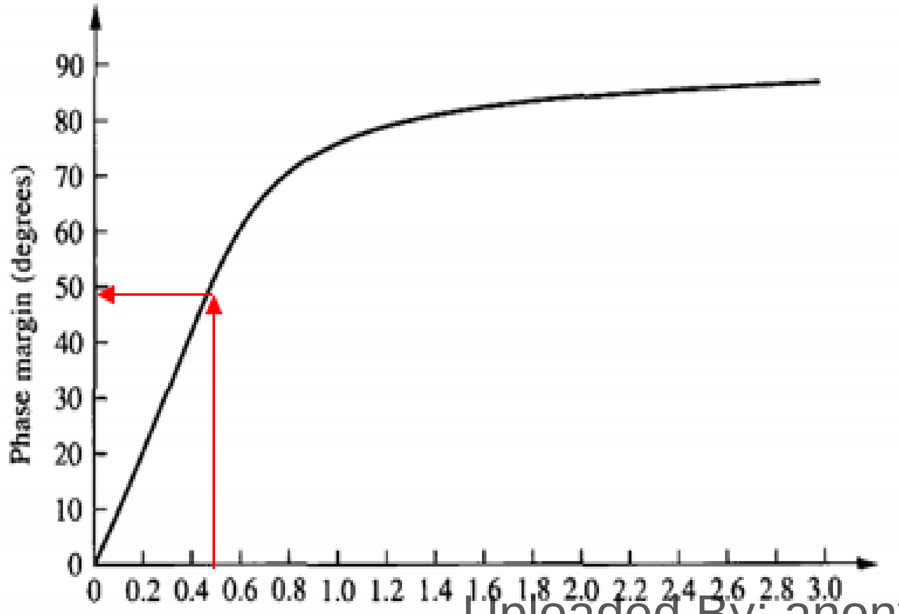
$$G(s) = \frac{600{,}000}{s(s+50)(s+120)}$$

# 4. Compute the Lead compensator phase margin $\phi_{max}$

- A 20% overshoot gives $\zeta$=0.456. Also the required phase margin PM=48.1°.
- For the uncompensated system with $K = 2$ the phase margin is 14.6° at a phase-margin frequency $\omega_{PM}$=58.4 rad/s. See the bode plots
- We need to add 5-12 to the phase margin to compensate the phase angle for the lead compensator.
- PM = PM+10 = 48.1+12 = 60.1
- The total phase contribution required from the compensator is:
  $\phi_{max}$= 60.1- 14.6= 45.5

- If the system's characteristics are not acceptable after the design, then a redesign with a different correction factor may be necessary.



## 5. Determine the value of $\beta$

$$\phi_{\max} = \tan^{-1}\frac{1-\beta}{2\sqrt{\beta}} = \sin^{-1}\frac{1-\beta}{1+\beta} \longrightarrow \beta = 0.18459 \ \ for\ \phi_{max} = 45.5$$

## 6. Determine the compensator's magnitude at $\omega_{max}$

$$|G_c(j\omega_{\max})| = \frac{1}{\sqrt{\beta}} \quad = 7.3379 \text{ dB}$$

## 7. Determine the new phase-margin frequency $\omega_{max}$.

- If we select $\omega_{max}$ to be the new phase-margin frequency, the uncompensated system's magnitude at this frequency must be -7.3379 to yield a 0 dB crossover at $\omega_{max}$ for the compensated system.

- The uncompensated system passes through 7.3379 dB at $\omega_{max}$ = 90.1 rad/s. This frequency is thus the new phase-margin frequency.

8. Design the lead compensator's break frequencies (T) fom (Eq. 11.9)

$$\omega_{max} = \frac{1}{T\sqrt{\beta}}$$ → for $\beta = 0.18459$ and $\omega_{max} = 90.1$ rad/sec

$$\frac{1}{T} = 38.7105 \quad , \qquad \frac{1}{\beta T} = 79.136$$

9. Hence, the compensator is given by:

$$C_c(s) = \frac{1}{\beta}\frac{(s+\frac{1}{T})}{(s+\frac{1}{\beta T})} = 5.4175\frac{(s+38.7105)}{(s+209.9772\ )}$$

10. Check the bandwidth for the system

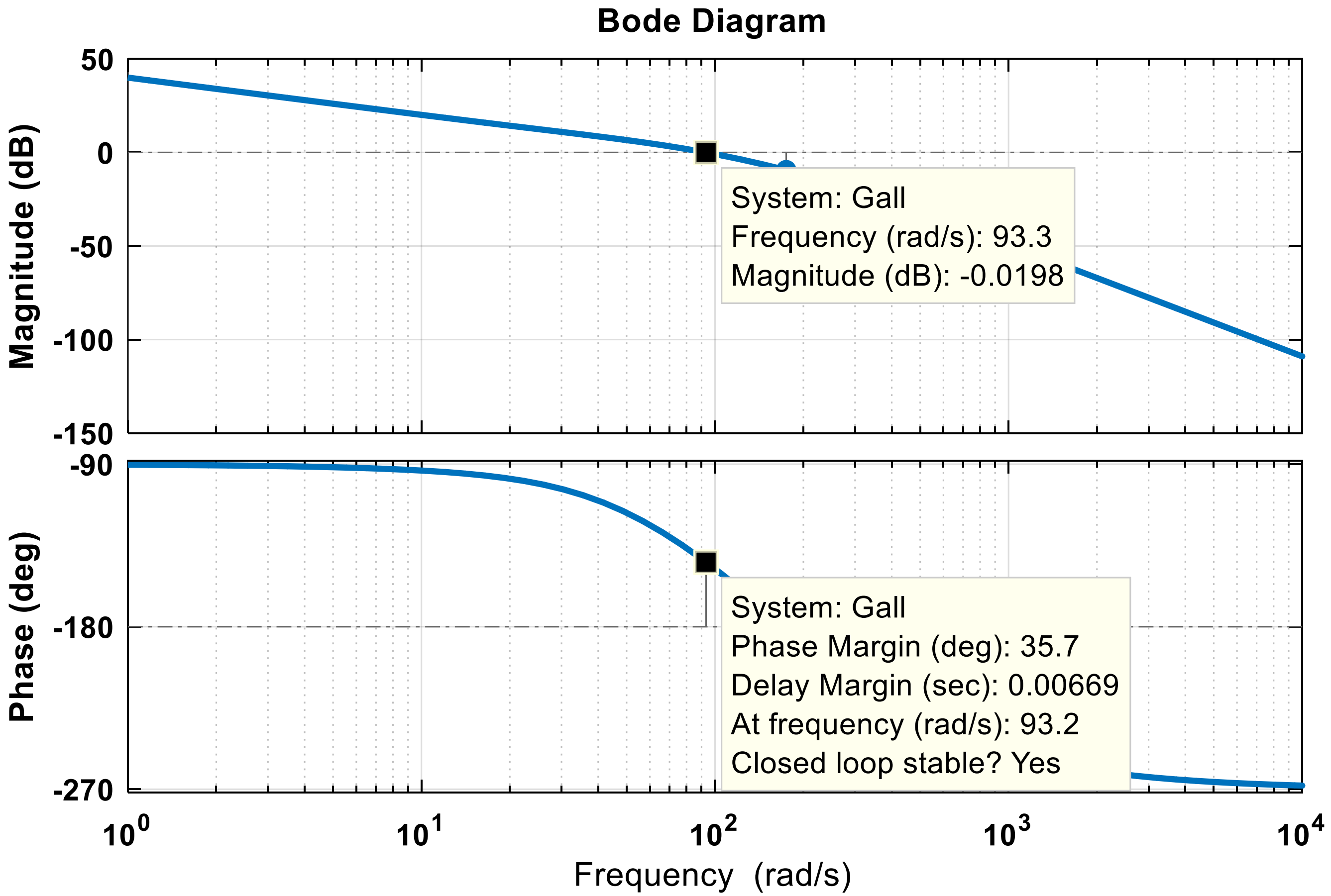
$$G_{all}(s) = C_c(s)G(s) = 5.4175\frac{(s+38.7105)}{(s+209.9772\ )} * \frac{600{,}000}{s(s+50)(s+120)}$$

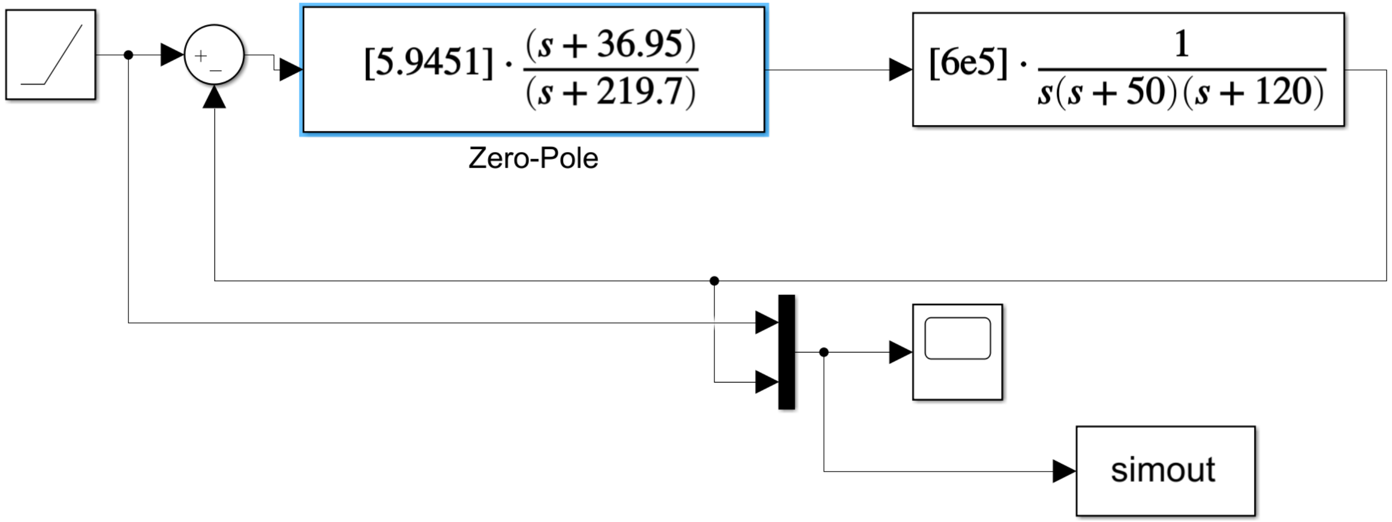
$$= 5.4175\frac{(s+38.7105)}{(s+209.9772\ )} * \frac{600000}{s(s+50)(s+120)}$$

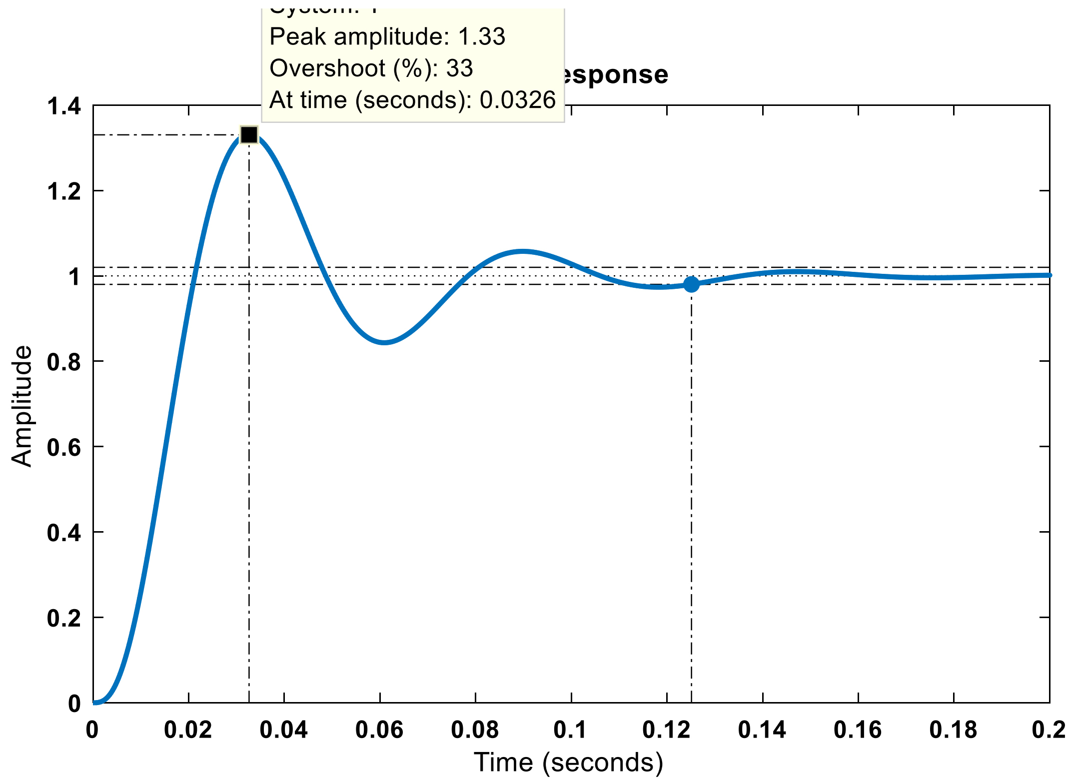
# 10. Check the bandwidth frequency $\omega_{BW}$

$$G_{all}(s)=5.4175\frac{(s+38.7105)}{(s+209.9772\ )}\ *\ \frac{600000}{s(s+50)(s+120)}$$

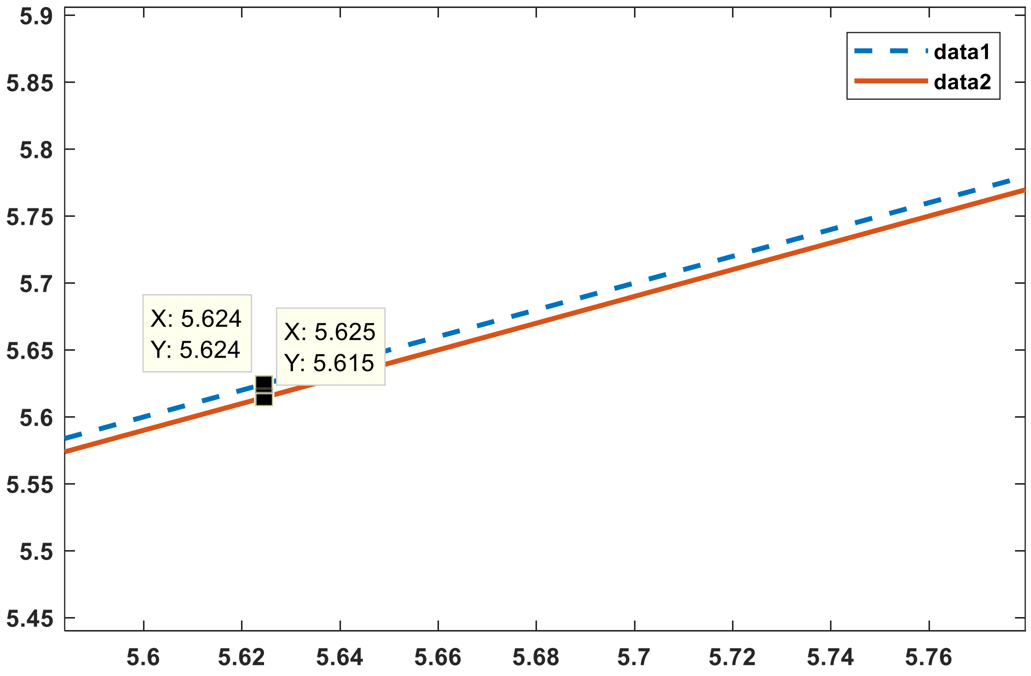




$[5.9451] \cdot \frac{(s+36.95)}{(s+219.7)}$
Zero-Pole
$[6e5] \cdot \frac{1}{s(s+50)(s+120)}$
simout

Peak amplitude: 1.33
Overshoot (%): 33
At time (seconds): 0.0326
esponse
Amplitude
Time (seconds)

data1
data2
X: 5.624
Y: 5.624
X: 5.625
Y: 5.615