4 Differentiation

الاشتقاق

Derivative مشتقة
Differentiate اشتق
Differentiable قابل للاشتقاق

4.1 Formal Definition of the Derivative

تعريف المشتقة

Definition The derivative of a function f at x, denoted by $\underbrace{f'(x)}_{f \text{ prime of } x}$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided that the limit exists.

Notation $y' = \frac{dy}{dx} = f'(x) = \frac{df}{dx} = \frac{d}{dx} f(x)$.

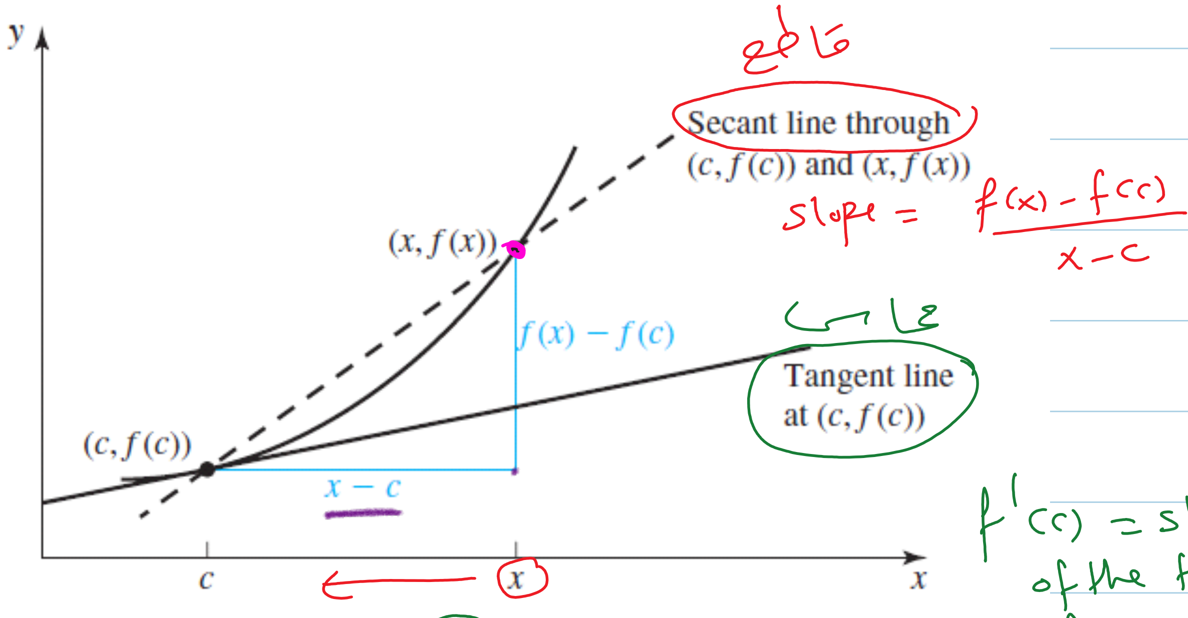


Figure 4.6 The derivative $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ is the slope of the tangent line at $(c, f(c))$.

$f'(c)$ = slope of the tangent line at $x = c$.

4.1.1 Geometric Interpretation and Using the Definition

Definition of the Tangent Line If the derivative of a function f exists at $x = c$, then the tangent line at $x = c$ is the line going through the point $(c, f(c))$ with slope

$$f'(c) = \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \text{slope of the tangent at } x = c.$$

Equation of the Tangent Line If the derivative of a function f exists at $x = c$, then $f'(c)$ is the slope of the tangent line at the point $(c, f(c))$. The equation of the tangent line is given by

$$\boxed{y - \underbrace{f(c)}_{y_1} = \underbrace{f'(c)}_{m}(x - \underbrace{c}_{x_1})}$$

slope $= m = f'(c)$

$(x_1, y_1) = (c, f(c))$

$y - y_1 = m(x - x_1)$

ex. If $f(x) = k$ "constant" then $f'(x) = 0$

Sol. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{k-k}{h}$ $\frac{0}{h}$ $h \neq 0$

$= \lim_{h\to 0} (0) = 0.$

ex. $f(x) = mx + b \Rightarrow f'(x) = m.$

Proof. $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h\to 0} \frac{[m(x+h)+b] - [mx+b]}{h}$

$= \lim_{h\to 0} \frac{\cancel{mx} + mh + \cancel{b} - \cancel{mx} - \cancel{b}}{h}$

$= \lim_{h\to 0} \frac{m\cancel{h}}{\cancel{h}} = \lim_{h\to 0} m = m.$

ex. $y = 1 + 2021x \Rightarrow y' = 2021.$

Sunday, July 04, 2021
9:12 PM

EXAMPLE 3 **Using the Definition** Find the derivative of

$x \neq 0 \Leftarrow \quad f(x) = \frac{1}{x} \quad \text{for } x \neq 0$

Sol.
$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$
$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{x-(x+h)}{x(x+h)}\right] \qquad \cancel{x} - \cancel{x} - h$$
$$= \lim_{h\to 0} \frac{1}{\cancel{h}}\left(\frac{-\cancel{h}}{x(x+h)}\right)$$
$$= \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x(x+0)} = \frac{-1}{x^2}, \quad x \neq 0.$$

ex. $f(x) = \sqrt{x}$, find, by def'n, $f'(9)$.

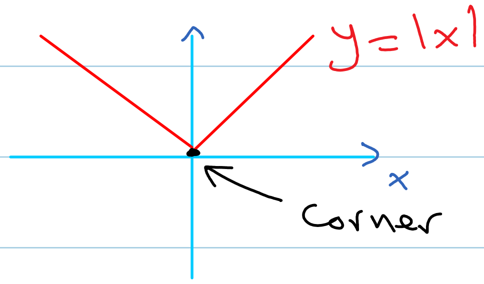
Sol.
$$f'(9) = \lim_{h\to 0} \frac{f(9+h) - f(9)}{h}$$
$$= \lim_{h\to 0} \left[\frac{\sqrt{9+h} - 3}{h} \cdot \frac{\sqrt{9+h} + 3}{\sqrt{9+h} + 3}\right]$$
$$= \lim_{h\to 0} \frac{\cancel{9} + h - \cancel{9}}{h\left(\sqrt{9+h} + 3\right)}$$
$$= \lim_{h\to 0} \frac{1}{\sqrt{9+h} + 3} = \frac{1}{\sqrt{9} + 3} = \frac{1}{6}.$$

4.1.2 (محذوف)

4.1.3 Differentiability and Continuity

Thm. If f is differentiable at $x=c$, then it is continuous at $x=c$, but the converse is not true

Ex. $y=|x|$



is cont. on $(-\infty,\infty)$

so it is cont. at $x=0$. but it is not diffble at $x=0$. Since

$$f(x)=\begin{cases} x, & x\geq 0 \\ -x, & x<0 \end{cases}$$

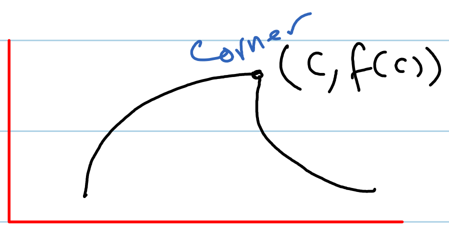
$|0|=0$

$$f'_+(0)=\lim_{\underbrace{h\to 0^+}_{h>0}}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{f(h)-f(0)}{h}$$

$$=\lim_{h\to 0^+}\frac{h-0}{h}=1$$

$$f'_-(0)=\lim_{h\to 0^-}\frac{f(h)-f(0)}{h}=\lim_{h\to 0}\frac{-h-0}{h}=-1$$

$$\Rightarrow f'_+(0)\neq f'_-(0)\Rightarrow f \text{ is not diffble at } x=0.$$



f is not diffble at $x=c$

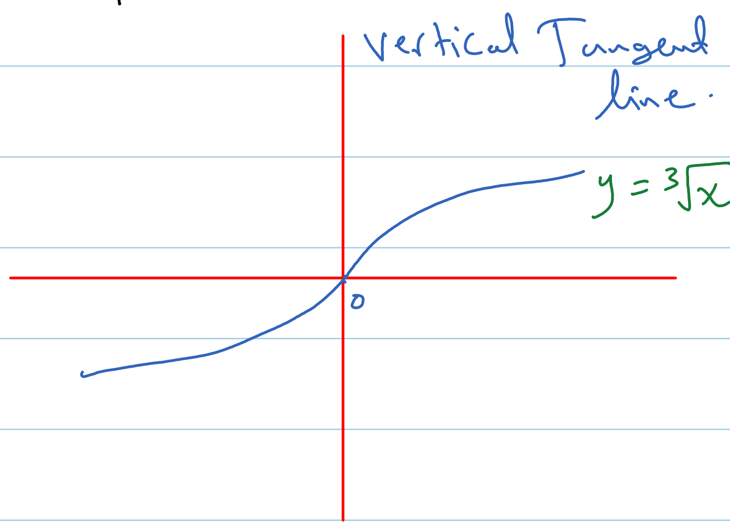
ex. $f(x) = x^{\frac{1}{3}}$ find $f'(0)$.

Sol. $f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$

$= \lim_{h \to 0} \frac{h^{\frac{1}{3}} - 0}{h^1}$

$= \lim_{h \to 0} \frac{1}{h^{\frac{2}{3}}} = +\infty$ (DNE)

$\therefore f'(0)$ DNE $\Rightarrow$ f is not diffble at $x=0$.



* If f is discontinuous at $x=c$, then it is not diffble at $x=c$

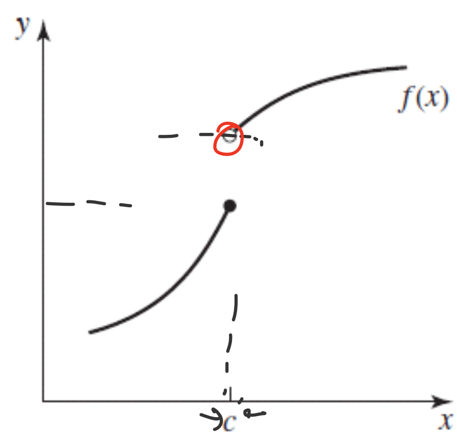


Figure 4.18 The function $y = f(x)$ is not differentiable at $x = c$.

f is not diffble at $x=c$ since f is discont. at $x=c$.

4.2 The Power Rule, the Basic Rules of Differentiation, and the Derivatives of Polynomials

① Power rule $\frac{d}{dx}(x^n) = n x^{n-1}$, $n \in \mathbb{R}$.

② $f(x) = \text{Constant} \Rightarrow f'(x) = 0$.

Ex. (a) $f(x) = x^6 \Rightarrow f'(x) = 6x^5$.

(b) $f(x) = x^{300} \Rightarrow f'(x) = 300\, x^{299}$

(c) $g(t) = t^5 \Rightarrow \frac{dg}{dt} = 5t^4$.

(d) $z = s^3$ $\quad \frac{dz}{ds} = 3s^2$

(e) $x = y^4$ $\quad \frac{dx}{dy} = 4y^3$.

(f) $f(x) = \dfrac{2021 + (1443)^{\frac{1}{5}}}{2\sqrt{6}} \Rightarrow f'(x) = 0$

$\boxed{3}$ $\frac{d}{dx}\left(a f(x)\right) = a \frac{df}{dx}$.

④ $\frac{d}{dx}\left(f(x) \pm g(x)\right) = f'(x) \pm g'(x)$.

ex. $y = 2x^4 - 3x^3 + x - 7$

Sol. $y' = 2(4x^3) - 3(3x^2) + 1$.

$= 8x^3 - 9x^2 + 1$.

Sunday, July 04, 2021
9:13 PM

ex. $\frac{d}{dx}(-5x^7+2x^3-10) = -5(7x^6)+2(3x^2)+0$
$= -35x^6+6x^2.$

ex. $\frac{d}{dt}(t^3-8t^2+3t) = 3t^2-16t+3.$

ex. $\frac{d}{dN}(\ln 5 + N\ln 7) = 0+\ln 7 = \ln 7.$

ex. $\frac{d}{dr}\left(r^2 \sin\frac{\pi}{4} - r^3\cos\frac{\pi}{12} + \sin\frac{\pi}{6}\right)$

$= 2r\sin\frac{\pi}{4} - 3r^2\cos\frac{\pi}{12}$

EXAMPLE 4 **Tangent and Normal Lines** If $f(x) = 2x^3 - 3x + 1$, find the tangent and normal lines at $(-1, 2)$.

Sol. $f'(x) = 2(3x^2) - 3 = 6x^2 - 3.$

Slope of the tangent line $= f'(-1) = 6(-1)^2 - 3 = 6 - 3 = \boxed{3}.$

Slope of the normal $= \frac{-1}{f'(-1)} = -\frac{1}{3}$

Tangent line $y - 2 = 3(x - -1)$

$\underset{+2}{y-2} = \underset{+2}{3x+3} \Rightarrow \boxed{y = 3x+5}$

Normal line $y - 2 = -\frac{1}{3}(x+1)$

$2 - \frac{1}{3}$ $\quad \underset{+2}{y-2} = -\frac{1}{3}x \underset{+2}{-\frac{1}{3}} \Rightarrow \boxed{y = -\frac{1}{3}x + \frac{5}{3}}$

4.3 The Product and Quotient Rules, and the Derivatives of Rational and Power Functions

4.3.1 The Product Rule

قاعدة الضرب

$$\boxed{h(x) = f(x)g(x) \Rightarrow h'(x) = f'(x)\,g(x) + f(x)\,g'(x)}$$

EXAMPLE 1 Differentiate $f(x) = (3x+1)(2x^2-5)$.

Sol.
$$f'(x) = (3x+1)'(2x^2-5) + (3x+1)(2x^2-5)'$$
$$= (3)(2x^2-5) + (3x+1)(4x)$$
$$= \underbrace{6x^2}-15 + \underbrace{12x^2}+4x$$
$$= 18x^2+4x-15.$$

Homework

EXAMPLE 2 Differentiate $f(x) = (3x^3-2x)^2$.

Sol.
$$f(x) = (3x^3-2x)(3x^3-2x)$$
$$f'(x) = (3x^3-2x)'(3x^3-2x) + (3x^3-2x)(3x^3-2x)'$$
$$= \underbrace{(9x^2-2)(3x^3-2x)} + \underbrace{(3x^3-2x)(9x^2-2)}$$
$$= 2(9x^2-2)(3x^3-2x).$$

Ingeneral, $(\underbrace{fg}h)' = (fg)(h') + (fg)'h$
$$= fgh' + (f'g + fg')h$$
$$= fgh' + f'gh + fg'h.$$

$$(fhgk)' = f'hgk + fh'gk + fhg'k + fhgk'.$$

Homework

EXAMPLE 4 Apply the product rule repeatedly to find the derivative of

$$y = (2x+1)(x+1)(3x-4)$$

Sol. $y' = (2x+1)'(x+1)(3x-4) + (2x+1)(x+1)'(3x-4) + (2x+1)(x+1)(3x-4)'$

$= 2(x+1)(3x-4) + (2x+1)(1)(3x-4) + 3(2x+1)(x+1)$

$= 2(3x^2-4x+3x-4) + (6x^2-8x+3x-4) + 3(2x^2+2x+x+1)$

$\vdots$

$= 18x^2+2x-9.$

4.3.2 The Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{g\cdot f' - f\cdot g'}{g^2}, \quad g \neq 0.$$

EXAMPLE 5 Differentiate $y = \frac{x^3-3x+2}{x^2+1}$. (This function is defined for all $x \in \mathbf{R}$, since $x^2+1 \neq 0$.)

$$y' = \frac{(x^2+1)(x^3-3x+2)' - (x^3-3x+2)(x^2+1)'}{(x^2+1)^2}$$

$$= \frac{(x^2+1)(3x^2-3) - (x^3-3x+2)(2x)}{(x^2+1)^2}$$

الاشتقاق الضمني

4.4.2 Implicit Functions and Implicit Differentiation

$$\boxed{y^5x^2 - yx + 2y^2 = \sqrt{x}}$$

y is implicitly a function of x.

$$\boxed{y = x^5 + x^2 + 1}$$

y is explicitly a function of x.

ex 12. Find $\frac{dy}{dx}$ if $x^2 + y^2 = 1$. $\quad (y(x))^2$

Sol. $\frac{d}{dx}(x^2) + \underbrace{\frac{d}{dx}(y^2)}_{\text{chain Rule}} = \frac{d}{dx}(1)$.

$$2x + 2y\frac{dy}{dx} = 0.$$

$$2y\frac{dy}{dx} = -2x \Rightarrow \frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

ex 13. Find $\frac{dy}{dx}$ if $y^3x^2 - yx + 2y^2 = x$.

Sol. $\frac{d}{dx}(y^3x^2) - \frac{d}{dx}(yx) + \frac{d}{dx}(2y^2) = \frac{d}{dx}(x)$.

$$y^3 \cdot 2x + x^2 \cdot 3y^2 y' - (y \cdot 1 + x \cdot y') + 4yy' = 1$$

$$3x^2y^2y' - xy' + 4yy' = 1 + y - 2xy^3.$$

$$(3x^2y^2 - x + 4y)y' = 1 + y - 2xy^3.$$

$$y' = \frac{1+y-2xy^3}{3y^2x^2-x+4y}.$$

Ex14. If $\boxed{y^2 = x^3, \; x>0, \; y>0}$. show that $y' = \frac{3}{2}\sqrt{x}$.

Sol.

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^3).$$

$$2yy' = 3x^2$$

$$\boxed{y' = \frac{3x^2}{2y}}$$

but $\sqrt{y^2} = \sqrt{x^3}$

$|y| = x^{\frac{3}{2}}$

$\boxed{y = x^{\frac{3}{2}}, \; y>0}$

$$= \frac{3x^2}{2x^{3/2}}$$

$$= \frac{3}{2}x^{2-\frac{3}{2}} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}$$

4.4 — 4.4.1, 4.4.2, 4.4.3, 4.4.4

4.4.3 Related Rates

EXAMPLE 15 Find $\frac{dy}{dt}$ when $x^2 + y^3 = 1$ and $\frac{dx}{dt} = 2$ for $x = \sqrt{7/8}$.

$y = y(t)$, $x = x(t)$.

Sol.

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^3) = \frac{d}{dt}(1)$$

$$\boxed{2x\frac{dx}{dt} + 3y^2\frac{dy}{dt} = 0}$$??

$x^2 + y^3 = 1$

$\left(\sqrt{\frac{7}{8}}\right)^2 + y^3 = 1$

$\frac{7}{8} + y^3 = 1$

$y^3 = \frac{1}{8}$

$\boxed{y = \frac{1}{2}}$

$$2\sqrt{\frac{7}{8}}(2) + 3\left(\frac{1}{2}\right)^2\frac{dy}{dt} = 0$$

$$4\sqrt{\frac{7}{8}} = -\frac{3}{4}\frac{dy}{dt}$$

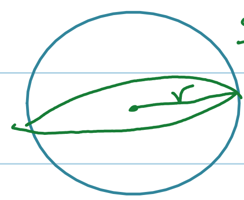
$$\therefore \frac{dy}{dt} = \left(4\sqrt{\frac{7}{8}}\right)\left(\frac{-4}{3}\right) = \frac{-16}{3}\sqrt{\frac{7}{8}} .$$

EXAMPLE 16 **Changing Volume** A spherical balloon is being filled with air. When the radius $r = 6$ cm, the radius is increasing at a rate of 2 cm/s. How fast is the volume changing at this time?

$\frac{dr}{dt} = 2$

حجم الكرة

Sol. $\boxed{V = \frac{4}{3}\pi r^3}$



Sphere

كرة

?? $\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \cdot \cancel{3}r^2\frac{dr}{dt}$

$$= 4\pi(6)^2(2) = 288\pi \text{ cm}^3/\text{s} .$$

61. Assume that x and y are differentiable functions of t. Find $\frac{dy}{dt}$ when $x^2 + y^2 = 1$, $\frac{dx}{dt} = 2$ for $x = \frac{1}{2}$, and $y > 0$.

$$\left(\tfrac{1}{2}\right)^2 + y^2 = 1$$

$$\tfrac{1}{4} + y^2 = 1$$

$$y^2 = \tfrac{3}{4}$$

$$y = \tfrac{\sqrt{3}}{2}$$

Sol.

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = \frac{d}{dt}(1)$$

??

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2\left(\tfrac{1}{2}\right)(2) + 2\left(\tfrac{\sqrt{3}}{2}\right)\frac{dy}{dt} = 0$$

$$2 = -\sqrt{3}\,\frac{dy}{dt} \Rightarrow \boxed{\frac{dy}{dt} = \frac{-2}{\sqrt{3}}}$$

Homework

66. Assume that the radius r and the area $A = \pi r^2$ of a circle are differentiable functions of t. Express dA/dt in terms of dr/dt.

Sol.

$$A = \pi r^2 \Rightarrow \boxed{\frac{dA}{dt} = 2\pi r\frac{dr}{dt}}$$

4.4.4 Higher Derivatives

المشتقات العليا.

$y = f(x)$, $y' = f'(x)$ first derivative

$\frac{d^2y}{dx^2} = y'' = f''(x) = f^{(2)}(x)$ second derv.

$\frac{d^3y}{dx^3} = y''' = f'''(x) = f^{(3)}(x)$ third derv.

$\vdots$

$\frac{d^ny}{dx^n} = y^{(n)} = f^{(n)}(x)$ nth derivative.

ex. $f(x) = x^5$, find $f^{(n)}(x)$, $n = 1, 2, \dots$

Sol. $\boxed{f'(x) = 5x^4}$

$\boxed{f''(x) = 20x^3}$, $\boxed{f'''(x) = 60x^2}$

$\boxed{f^{(4)}(x) = 120x}$

$\boxed{f^{(5)}(x) = 120}$, $\boxed{f^{(6)}(x) = 0}$

$\vdots$

$\boxed{f^{(n)}(x) = 0}$, $n = 6, 7, 8, \dots\dots$

ex. $y = \sqrt{x}$ find $y^{(2)}$.

Sol. $y' = \dfrac{1}{2\sqrt{x}} = \dfrac{1}{2} x^{-\frac{1}{2}} \Rightarrow y'' = \dfrac{1}{2}\left(-\dfrac{1}{2}\right) x^{-\frac{3}{2}}$

$= -\dfrac{1}{4} x^{-\frac{3}{2}}$

ex. Find $\dfrac{d^3y}{dx^3}$ if $\boxed{x^2 + y^2 = 1}$.

Sol. $\dfrac{d}{dx}(x^2) + \dfrac{d}{dx}(y^2) = \dfrac{d}{dx}(1)$

$2x + 2yy' = 0 \Rightarrow \boxed{y' = -\dfrac{x}{y}}$

$y'' = \dfrac{y \cdot (-1) - (-x) \cdot y'}{y^2}$

$= \dfrac{-y + x \cdot \frac{-x}{y}}{y^2}$

من المعادلة الأصلية

$= \dfrac{-(y^2 + x^2)}{y^3} = -\dfrac{1}{y^3}$

$$y'' = -y^{-3}$$

$$y''' = -(-3)\,y^{-4}\,y'$$

$$= 3\,y^{-4}\cdot\left(\frac{-x}{y}\right) = \frac{-3x}{y^5}.$$

موقع

EXAMPLE 21 **Acceleration** Assume that the position of a car moving along a straight line is given by

$$\boxed{s(t) = 3t^3 - 2t + 1}$$

Find the car's velocity and acceleration.

Sol. $V(t) = s'(t) = 3(3t^2) - 2 = 9t^2 - 2.$

$a(t) = s''(t) = 9(2t) = 18t.$

jerk $j(t) = s'''(t) = 18.$

4.5 Derivatives of Trigonometric Functions

حفظ

① $\dfrac{d}{dx}\sin x = \cos x$ ② $\dfrac{d}{dx}\cos x = -\sin x$

③ $\dfrac{d}{dx}\tan x = \sec^2 x$ ④ $\dfrac{d}{dx}\cot x = -\csc^2 x$

⑤ $\dfrac{d}{dx}\sec x = \sec x \tan x$ ⑥ $\dfrac{d}{dx}\csc x = -\csc x \cot x$

Pf. $\dfrac{d}{dx}\tan x = \dfrac{d}{dx}\left(\dfrac{\sin x}{\cos x}\right) = \dfrac{(\cos x)(\cos x) - (\sin x)(-\sin x)}{\cos^2 x}$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x.$$

⑤ $\dfrac{d}{dx}\sec x = \dfrac{d}{dx}\left(\dfrac{1}{\cos x}\right) = \dfrac{d}{dx}[\cos x]^{-1}$

$$= -1(\cos x)^{-2}\cdot(-\sin x)$$

$$= \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x \tan x.$$

EXAMPLE 1 Find the derivative of $f(x) = -4\sin x + \cos\frac{\pi}{6}$.

Constant

Sol. $f'(x) = -4\cos x + 0 = -4\cos x.$

EXAMPLE 2 Find the derivative of $y = \cos(x^2 + 1)$.

$y' = -\sin(x^2+1)\cdot(x^2+1)' = -\sin(x^2+1)(2x)$

$= -2x\sin(x^2+1).$

EXAMPLE 3 Find the derivative of $y = \underbrace{x^2 \cdot \sin(3x)} - \cos(5x)$.

Sol. $y' = (x^2)(\cos(3x)\cdot 3) + (\sin 3x)(2x) + \sin(5x).5$

$= 3x^2 \cos(3x) + 2x \sin(3x) + 5\sin(5x).$

EXAMPLE 4 Compare the derivatives of

(a) $\tan x^2$ **(b)** $\tan^2 x$

Sol. (a) $y = \tan x^2 = \tan(x^2)$

$y' = \sec^2(x^2) \cdot 2x = 2x \sec^2(x^2).$

(b) $y = \tan^2 x = (\tan x)^2$

$y' = 2(\tan x)^1 \cdot \sec^2 x = 2\tan x \cdot \sec^2 x.$

EXAMPLE 5 **Repeated Application of the Chain Rule** Find the derivative of $f(x) = \sec(\sqrt{x^2+1})$.

Sol. $f(x) = \sec(\sqrt{x^2+1})$.

$f'(x) = \sec(\sqrt{x^2+1}) \tan(\sqrt{x^2+1}) \cdot (\sqrt{x^2+1})'$

$= \frac{x}{\sqrt{x^2+1}} \sec(\sqrt{x^2+1}) \tan(\sqrt{x^2+1}).$

29. $f(x) = \sqrt{\sin(2x^2-1)}$

$f'(x) = \frac{\cos(2x^2-1) \cdot 4x}{2\sqrt{\sin(2x^2-1)}} = \frac{2x \cos(2x^2-1)}{\sqrt{\sin(2x^2-1)}}.$

47. $g(x) = \dfrac{1}{\csc^3(1-5x^2)} = \sin^3(1-5x^2)$.

$= \left(\sin(1-5x^2)\right)^3$

$g'(x) = 3\left(\sin(1-5x^2)\right)^2 \cdot \cos(1-5x^2) \cdot (-10x)$.

$= -30x \sin^2(1-5x^2) \cos(1-5x^2)$.

57. $f(x) = \dfrac{\sec x^2}{\sec^2 x} = \sec(x^2) \cdot (\cos x)^2$

$y' = \left(\sec(x^2)\right)\left(2(\cos x)^1(-\sin x)\right)$

$+ (\cos x)^2 \left(\sec x^2 \tan x^2\right) \cdot 2x$.

$= \ldots$

60. Find the points on the curve $y = \cos^2 x$ that have a horizontal tangent.

تذكر

$\boxed{\sin(2A) = 2\sin A \cos A}$

Sol. $y = (\cos x)^2$

$y' = 2(\cos x)(-\sin x) = 0$.

$\Rightarrow -\sin(2x) = 0$

$\Rightarrow \boxed{\sin(2x) = 0}$

$\boxed{\sin A = 0}$ iff $A = n\pi$

$n = 0, \pm1, \pm2, \ldots$

$\boxed{y = \frac{1}{x}}$

$y^{(n)} = ??$

$2x = n\pi, \quad n = 0, \pm1, \pm2, \ldots$

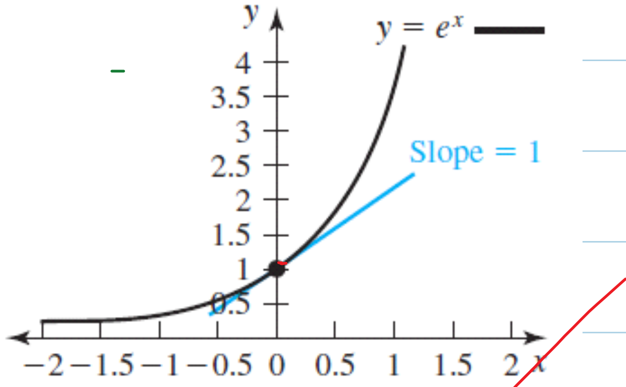
$\boxed{x = \frac{n\pi}{2}}, \quad n = 0, \pm1, \pm2, \ldots$

Points: $\left(\frac{n\pi}{2}, \cos^2\left(\frac{n\pi}{2}\right)\right), \quad n = 0, \pm1, \pm2, \ldots$

4.6 Derivatives of Exponential Functions

$$\boxed{\lim_{h\to 0}\frac{e^h-1}{h}=1} \quad \left(\frac{0}{0}\right) \quad \to \ln e = 1$$

h	0.1	0.01	0.001	0.0001
$\frac{e^h-1}{h}$	1.0517	1.0050	1.00050	1.000050



$$\lim_{h\to 0}\frac{a^h-1}{h}=\boxed{\ln a}$$

$$\boxed{\frac{d}{dx}e^x=e^x.}$$

$$\boxed{\frac{d}{dx}a^x=(\ln a)a^x}$$

$$\boxed{\frac{d}{dx}e^{g(x)}=g'(x)e^{g(x)}.}$$

$$\boxed{\frac{d}{dx}\left[a^{g(x)}\right]=\underbrace{a^{g(x)}}_{\text{نفسه}}\cdot\underbrace{g'(x)}_{\text{مشتقة الأس}}\cdot\underbrace{\ln a}_{\text{Ln الأساس}}} \quad (*)$$

$$\frac{d}{dx}(a^x)=\left[e^{\ln(a^x)}\right]'$$

$$\boxed{e^{\ln A}=A}, \ A>0.$$

$$=\left(e^{\overset{g(x)}{x\ln a}}\right)'=e^{x\ln a}\cdot(x\ln a)'$$

$$=e^{\ln(a^x)}\cdot(\ln a)=a^x(\ln a)$$

$$\Rightarrow \frac{d}{dx}(a^x)=a^x(\ln a)$$

EXAMPLE 1 Find the derivative of $f(x)=e^{-x^2/2}$.

$$f'(x)=e^{-\frac{x^2}{2}}\cdot\left(-\frac{x^2}{2}\right)'\cdot\ln e$$

$$=e^{-\frac{x^2}{2}}\left(-\frac{\cancel{2}x}{\cancel{2}}\right)\cdot 1=-x\,e^{-\frac{x^2}{2}}.$$

Ex2 $y=3^{\sqrt{x}} \Rightarrow y'=3^{\sqrt{x}}\cdot\left(\frac{1}{2\sqrt{x}}\right)(\ln 3) = \frac{\ln 3}{2}\,\frac{3^{\sqrt{x}}}{\sqrt{x}}.$

ex3. $y = e^{\sin(\sqrt{x})}$

$y' = e^{\sin(\sqrt{x})} \cdot (\sin\sqrt{x})' \cdot (\ln e)$

$= e^{\sin(\sqrt{x})} \cdot \cos(\sqrt{x}) \cdot \frac{1}{2\sqrt{x}}$.

ex4. Find $\lim_{h\to 0} \frac{3^{2h} - 1}{h - 0}$ $f(h)$, $f(0)$ $1 = 3^{2(0)} = f(0)$

$= \lim_{h\to 0} \frac{f(h) - f(0)}{h - 0} = f'(0)$, $f(x) = 3^{2x}$

$f'(x) = 3^{2x} \cdot (2) \ln 3$

$= 3^0 (2) \ln 3$

$= 2\ln 3 = \ln 3^2 = \ln 9$.

Q53) $\lim_{h\to 0} \frac{\overset{f(h)}{e^{2h}} - \overset{f(0)}{1}}{h - 0} = f'(0)$ where $f(x) = e^{2x}$

$= 2e^0 = 2(1)$

$= 2$.

$f'(x) = e^{2x} \cdot (2) \ln e$

$= 2e^{2x}$.

Q54) $\lim_{h\to 0} \frac{\overset{f(h)}{e^{5h}} - \overset{f(0)}{1}}{3h} = \frac{1}{3} \lim_{h\to 0} \frac{f(h) - f(0)}{h - 0}$

$= \frac{1}{3} f'(0)$, where $f(x) = e^{5x}$.

$f'(x) = 5e^{5x}$

$= \frac{1}{3}(5e^0)$

$= \frac{5}{3}$.

20) $f(x) = e^{\cos(1-2x^3)}$

$f'(x) = e^{\cos(1-2x^3)} \cdot (-\sin(1-2x^3)) \cdot (-6x^2)$.

48) $h(t) = 6^{\sqrt{6t^6 - 6}} \Rightarrow h'(t) = 6^{\sqrt{6t^6-6}} \cdot \left(\frac{36t^5}{2\sqrt{6t^6-6}}\right) (\ln 6)$

4.7.1 4.7.2 4.7.3

4.7 Derivatives of Inverse Functions, Logarithmic Functions, and the Inverse Tangent Function

4.7.1 Derivatives of Inverse Functions

Derivative of an Inverse Function If $f(x)$ is one to one and differentiable with inverse function $f^{-1}(x)$ and $f'[f^{-1}(x)] \neq 0$, then $f^{-1}(x)$ is differentiable and

$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'[f^{-1}(x)]} \tag{4.12}$$

$$\left.\frac{d}{dx} f^{-1}(x)\right|_{\boxed{x=b}} = \frac{1}{\left.\frac{df}{dx}\right|_{\boxed{x=f^{-1}(b)}}}$$

$y = f(x)$

$b = f(x)$

$\vdots$

$x = ??$

Ex ① . $\boxed{f(x) = \frac{x}{1+x}}$, $x \geq 0$. Find $\left.\frac{df^{-1}}{dx}\right|_{x=\frac{1}{3}}$.

Sol.

$$\boxed{\left.\frac{df^{-1}}{dx}\right|_{\boxed{x=\frac{1}{3}}} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}\left(\frac{1}{3}\right)}}}$$

$f^{-1}\left(\frac{1}{3}\right) = ??$

$\frac{1}{3} = \frac{x}{1+x}$

$3x = 1 + x$

$2x = 1$

$x = \frac{1}{2} = f^{-1}\left(\frac{1}{3}\right)$

$$= \frac{1}{f'\left(\frac{1}{2}\right)}$$

$$f'(x) = \frac{(1+x)(1) - x(1)}{(1+x)^2} = \frac{1}{(1+x)^2}$$

$$f'\left(\tfrac{1}{2}\right) = \frac{1}{\left(1+\frac{1}{2}\right)^2} = \frac{1}{9/4} = \frac{4}{9}.$$

$$\therefore \left.\frac{df^{-1}}{dx}\right|_{x=\frac{1}{3}} = \frac{1}{f'\left(\frac{1}{2}\right)} = \frac{9}{4}.$$

ex (2). $\boxed{f(x) = 2x + e^x}$. Find $\left.\frac{df^{-1}}{dx}\right|_{x=1}$

Sol. $\left.\frac{df^{-1}}{dx}\right|_{x=1} = \dfrac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(1)}}$ ✓

$f^{-1}(1) = ??$

$\boxed{1 = 2x + e^x}$ ✓

by inspection
تخمين

$x = 0 = f^{-1}(1)$

$= \dfrac{1}{f'(0)}$

$= \left.\dfrac{1}{2+e^x}\right|_{x=0}$

$= \dfrac{1}{2+1} = \dfrac{1}{3}$

ex (3) $f(x) = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$

Find $\left.\frac{df^{-1}}{dx}\right|_{x=1}$.

Sol. $\boxed{f'(x) = \sec^2 x}$.

$f^{-1}(1) = ??$

$\tan x = 1$

$x = \frac{\pi}{4} = f^{-1}(1)$

$\left.\frac{df^{-1}}{dx}\right|_{x=1} = \dfrac{1}{f'(\frac{\pi}{4})} = \dfrac{1}{\sec^2(\frac{\pi}{4})} = \dfrac{1}{2}$

16. Let

$$f(x) = x^2 + \tan x, \quad x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \text{Find } \left.\frac{d}{dx} f^{-1}(x)\right|_{x=\frac{\pi^2}{16}+1}$$

Sol. $f'(x) = 2x + \sec^2 x$.

$f^{-1}\left(\frac{\pi^2}{16}+1\right) = ??$

$\frac{\pi^2}{16} + 1 = x^2 + \tan x$.

$\left(\frac{\pi}{4}\right)^2 + 1 = x^2 + \tan x$ ✓

$x = \frac{\pi}{4}$ بالتخمين.

$\left.\frac{df^{-1}}{dx}\right|_{x=\frac{\pi^2}{16}+1} = \dfrac{1}{f'(\frac{\pi}{4})}$

$= \dfrac{1}{2\frac{\pi}{4} + \sec^2(\frac{\pi}{4})}$

$$= \frac{1}{\frac{\pi}{2}+2} = \frac{2}{\pi+4}.$$

If $f(x) = \tan x \Rightarrow \dfrac{df^{-1}}{dx} = \boxed{\dfrac{d}{dx}\left(\underbrace{\tan^{-1} x}_{\arctan x}\right) = \dfrac{1}{1+x^2}}$ $x \in \mathbb{R}$

If $f(x) = \sin x \Rightarrow \boxed{\dfrac{df^{-1}}{dx} = \dfrac{d}{dx}\underbrace{\sin^{-1} x}_{\arcsin x} = \dfrac{1}{\sqrt{1-x^2}}}$

$-1 < x < 1$

ex. $y = \sin^{-1}(x^3)$

$$y' = \frac{1}{\sqrt{1-(x^3)^2}} \cdot (x^3)' = \frac{3x^2}{\sqrt{1-x^6}}.$$

ex. $y = e^{\tan^{-1} x} \Rightarrow y' = e^{\tan^{-1} x} \cdot \left(\dfrac{1}{1+x^2}\right)$

ex. $y = \tan^{-1}(\sqrt{x})$

$$y' = \frac{1}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}(1+x)}$$

4.7.2 The Derivative of the Logarithmic Function

$f(x) = e^x$, $f'(x) = e^x$ and $f^{-1}(x) = \ln x$.

$$\frac{d}{dx}\ln x = \frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}$$

$$= \frac{1}{e^{f^{-1}(x)}} = \frac{1}{e^{\ln x}} = \frac{1}{x}$$

$$\Rightarrow \boxed{\frac{d}{dx}(\ln x) = \frac{1}{x}}$$

$$\boxed{\frac{d}{dx}(\ln g(x)) = \frac{g'(x)}{g(x)}}$$

$$\frac{d}{dx}\left(\log_a g(x)\right) = \frac{d}{dx}\left(\frac{\ln g(x)}{\ln a}\right)$$

$$= \frac{1}{\ln a}\frac{d}{dx}(\ln g(x))$$

In general,

$$\boxed{\frac{d}{dx}\log_a(g(x)) = \frac{1}{\ln a}\,\frac{g'(x)}{g(x)}}$$

ex. $y = \ln(3x) \Rightarrow y' = \dfrac{(3x)'}{3x} = \dfrac{3}{3x} = \dfrac{1}{x}$.

ex: $y = \ln(x^2+1) \Rightarrow y' = \dfrac{2x}{x^2+1}$

ex. $y = \ln(\sin x) \Rightarrow y' = \dfrac{\cos x}{\sin x} = \cot x$.

Sunday, July 04, 2021
9:12 PM

ex. $y = \log(2x^3-1) = \log_{10}(2x^3-1)$

$$y' = \frac{1}{\ln 10} \cdot \frac{(2x^3-1)'}{2x^3-1} = \frac{1}{\ln 10} \cdot \frac{6x^2}{2x^3-1}.$$

Q35) $f(x) = \ln(\sqrt{x^2+1}) = \ln(x^2+1)^{\frac{1}{2}} = \frac{1}{2}\ln(x^2+1).$

$$f'(x) = \frac{1}{2} \cdot \frac{2x}{x^2+1} = \frac{x}{x^2+1}.$$

Q54) $f(x) = \log_{10}(2x^2-1)$

$$f'(x) = \frac{1}{\ln 10} \cdot \frac{4x}{2x^2-1}.$$

59) $f(u) = \log_3(3+u^4)$

$$f'(u) = \frac{1}{\ln 3} \cdot \frac{4u^3}{3+u^4}.$$

تذكر

$$\frac{d\sin^{-1}x}{dx} = \frac{1}{\sqrt{1-x^2}}$$

ex. $f(x) = \log_2[\sin^{-1}(x^2)]$

$$f'(x) = \frac{1}{\ln 2} \cdot \frac{\frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x}{\sin^{-1}(x^2)}$$

$$= \frac{2x}{(\ln 2)\sin^{-1}(x^2)\sqrt{1-x^4}}$$

Rmk: $\sin^{-1}x \neq (\sin x)^{-1}$

$\sin^{-1}x$ is sine inverse.

$(\sin x)^{-1} = \frac{1}{\sin x} = \csc x$

In general, $f^{-1}(x) \neq \frac{1}{f(x)}$

f^{-1} ↓ the inverse of f.

STUDENTS-HUB.com

Example: Let $f(x) = x^x$ find $f'(x)$

$\Rightarrow \quad y = x^x$ "important"

$\Rightarrow \ln y = \ln(x^x)$

$\Rightarrow \ln y = x \ln x$

$\Rightarrow \dfrac{dy}{y} = \dfrac{x}{x} + \ln x$

$\Rightarrow \dfrac{dy}{y} = 1 + \ln x$

$\Rightarrow dy = (1 + \ln x) \cdot y$

$= (1 + \ln x) \cdot x^x.$