3.2 Subspaces

* Given a vector space V, we can form another vector space $S \subseteq V$. using the operations of V.

Def If S is a nonempty subset of a vector space V s.t

$\boxed{1}$ $\alpha \vec{x} \in S$ for any $\vec{x} \in S$ and any scalar α

$\boxed{2}$ $\vec{x} + \vec{y} \in S$ for any $\vec{x} \in S$ and $\vec{y} \in S$,

then S is a subspace of V.

* That is, S must be closed under the operations of addition and scalar multiplication.

Exp Let $S = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_2 = 2x_1 \right\}$. Is S is a subspace of $\mathbb{R}^2$.

Yes since • S is nonempty. That is $\vec{x} = (0,0) \in S$, then

• If $\vec{x} = (a, 2a)$ is any vector in S, then

$\alpha \vec{x} = (\alpha a, \alpha(2a)) = (\alpha a, 2(\alpha a)) \in S$

• If $\vec{x} = (a, 2a)$ and $\vec{y} = (b, 2b)$ are arbitrary elements of S, then

$\vec{x} + \vec{y} = (a+b, 2a+2b) = (a+b, 2(a+b)) \in S$

Hence, S is a subspace of $\mathbb{R}^2$

Remarks $\boxed{1}$ Every subspace of a vector space is a vector space in its own right.

$\Rightarrow$ since it satisfies the 8 axioms of a vector space.

$\boxed{2}$ If V is a vector space, then $\{0\}$ and V are subspaces of V. All other subspaces are called proper subspaces.

$\Rightarrow \{0\}$ is called the zero subspace

$\boxed{3}$ Every subspace must contain the zero vector.

So we can show that S is nonempty by showing that $\vec{0} \in S$.

Exp show that the set $S = \{(x_1, x_2, x_3)^T \mid x_1 = x_2\}$ is a subspace of $\mathbb{R}^3$.

- S is nonempty since $\vec{0} = (0,0,0) \in S$.
- If $x = (a,a,b)^T$ is any vector in S, then

$$\alpha x = (\alpha a, \alpha a, \alpha b)^T \in S$$

- If $x = (a,a,b)^T$ and $y = (c,c,d)^T$ are arbitrary elements of S, then

$$x + y = (a+c,\ a+c,\ b+d)^T \in S$$

Exp Is $S = \left\{ \begin{pmatrix} x \\ 1 \end{pmatrix} : x \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^2$

No since $\begin{pmatrix} 2 \\ 1 \end{pmatrix} \in S$ but $\alpha \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\alpha \\ \alpha \end{pmatrix} \notin S$ when $\alpha \neq 1$

Exp show that $S = \{A \in \mathbb{R}^{2\times 2} : a_{12} = -a_{21}\}$ is a subspace of $\mathbb{R}^{2\times 2}$

- S is nonempty since $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in S$ or $\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} \in S$
- If $A = \begin{bmatrix} a & b \\ -b & c \end{bmatrix}$ is any matrix in S, then

$$\alpha A = \begin{bmatrix} \alpha a & \alpha b \\ -\alpha b & \alpha c \end{bmatrix} \in S$$

- If $A_1 = \begin{bmatrix} a_1 & b_1 \\ -b_1 & c_1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} a_2 & b_2 \\ -b_2 & c_2 \end{bmatrix}$ are arbitrary elements of S, then $A_1 + A_2 = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ -(b_1+b_2) & c_1 + c_2 \end{bmatrix} \in S$.

Notes ① P_n : is the set of all polynomials of degree less than n.

$$\Rightarrow P_2 = \{p(x) : p(x) = ax + b\}$$

$$\Rightarrow P_3 = \{p(x) : p(x) = ax^2 + bx + c\}$$

② $C^n[a,b]$: is the set of all functions f that have continuous n^{th} derivative on $[a,b]$.

$$\Rightarrow C^2[a,b] = \{f(x) : f'' \text{ is continuous on } [a,b]\}$$

Exp Show that $S = \{p(x) \in P_n : p(0) = 0\}$ is a subspace of P_n.

- S is not empty since S contains the zero polynomial
  $$z(x) = 0x^{n-1} + 0x^{n-2} + \cdots + 0 = 0$$
- If $p(x)$ is any polynomial in S, then
  $\alpha p(0) = \alpha(0) = 0$. Hence $\alpha p(x) \in S$.
- If $p(x)$ and $q(x)$ are arbitrary polynomials in S, then
  $(p+q)(0) = p(0) + q(0) = 0 + 0 = 0$. Hence $p + q \in S$.

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Exp Show that $C^n[a,b]$ is a subspace of $C[a,b]$.

- $C^n[a,b]$ is nonempty since $z(x) \in C^n[a,b]$
- If $f(x) \in C^n[a,b]$, then $f^{(n)}(x)$ is continuous on $[a,b]$.
  $(\alpha f)^{(n)}(x) = \alpha f^{(n)}(x)$ is continuous on $[a,b]$. scalar multiple of continuous function is continuous.
  Hence, $\alpha f \in C^n[a,b]$.
- If f and g are arbitrary elements of $C^n[a,b]$, then both have n^{th} derivatives that are continuous. Hence, their sum will have a continuous n^{th} derivative.
  Thus, $f+g \in C^n[a,b]$.

Exp Show that $S = \{f \in C^2[a,b] : f''(x) + f(x) = 0 \ \forall x \in [a,b]\}$ is a subspace of $C^2[a,b]$

- S is not empty since $z(x) \in S$ "zero function is in S"
- If f is any element in S, then $\forall x \in [a,b]$
  $(\alpha f)''(x) + (\alpha f)(x) = \alpha f''(x) + \alpha f(x) = \alpha[f''(x) + f(x)] = \alpha(0) = 0$
  Thus, $\alpha f \in S$.
- If f and g are arbitrary elements in S, then
  $$\begin{aligned}(f+g)''(x) + (f+g)(x) &= f''(x) + g''(x) + f(x) + g(x) \\ &= (f''(x) + f(x)) + (g''(x) + g(x)) \\ &= 0 + 0 = 0.\end{aligned}$$
  Thus, $f+g \in S$.

Exp Let S be the set of all 2×2 triangular matrices. Is S a subspace of $\mathbb{R}^{2\times 2}$.

No since $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}\in S$ and $B=\begin{bmatrix}0&0\\1&0\end{bmatrix}\in S$ but

$$A+B=\begin{bmatrix}0&1\\1&0\end{bmatrix}\notin S.$$

Def Let A be $m\times n$ matrix. The Null Space of A is the set of all solutions of the homogenuous system $Ax=0$.

That is, $N(A)=\{x\in\mathbb{R}^n \mid Ax=0\}$

Exp Show that $N(A)$ is a subspace of $\mathbb{R}^n$.

- $N(A)$ is non empty since $\vec{0}\in\mathbb{R}^n$
- If $\vec{x}$ is any vector in $N(A)$, then
  $A(\alpha\vec{x})=\alpha A\vec{x}=\alpha(\vec{0})=\vec{0}$. Hence $\alpha\vec{x}\in N(A)$.
- If $\vec{x}$ and $\vec{y}$ are arbitrary elements of $N(A)$, then
  $A(\vec{x}+\vec{y})=A\vec{x}+A\vec{y}=\vec{0}+\vec{0}=\vec{0}$. Hence, $\vec{x}+\vec{y}\in N(A)$

Exp* Find $N(A)$ if $A=\begin{bmatrix}1&1&1&0\\2&1&0&1\end{bmatrix}$

Recall that $N(A)=\{\vec{x}\in\mathbb{R}^n : A\vec{x}=\vec{0}\}$. $Ax=0 \Rightarrow$

$$\begin{bmatrix}1&1&1&0\\2&1&0&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$ We use Gauss-Jordan Reduction to solve this system.

The augmented matrix is

$$\left[\begin{array}{cccc|c}1&1&1&0&0\\\boxed{2}&1&0&1&0\end{array}\right]\xrightarrow{R_2-2R_1}\left[\begin{array}{cccc|c}1&\boxed{1}&1&0&0\\0&-1&-2&1&0\end{array}\right]\xrightarrow{R_1+R_2}$$

$$\left[\begin{array}{cccc|c}1&0&-1&1&0\\0&-1&-2&1&0\end{array}\right]\xrightarrow{-R_2}\left[\begin{array}{cccc|c}1&0&-1&1&0\\0&1&2&-1&0\end{array}\right]$$

Let $x_3=\alpha$ and $x_4=\beta$. Hence, $x_1=\alpha-\beta$ and $x_2=\beta-2\alpha$

$$x=\begin{pmatrix}\alpha-\beta\\-2\alpha+\beta\\\alpha\\\beta\end{pmatrix}=\alpha\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}+\beta\begin{pmatrix}-1\\1\\0\\1\end{pmatrix}$$ is a solution of $Ax=0$. $\Longrightarrow$

The vector space $N(A) = \left\{ X \in \mathbb{R}^4 : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \ \alpha, \beta \in \mathbb{R} \right\}$

Def. Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ be vectors in a vector space V.

- The sum $\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_n \vec{v}_n$ is called a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$, where $\alpha_1, \alpha_2, \dots, \alpha_n$ are scalars.
- The set of all linear combinations of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ is called the span of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ denoted by $\text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$.

Note: In Exp*: $N(A) = \text{span}\left((1,-2,1,0)^T, (-1,1,0,1)^T\right)$

Exp: In $\mathbb{R}^3$:

$\boxed{1}$ $\text{span}(e_1, e_2) = \left\{ v \in \mathbb{R}^3 : v = \alpha_1 e_1 + \alpha_2 e_2 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ 0 \end{pmatrix} \right\}$

$\boxed{2}$ $\text{span}(e_1, e_2, e_3) = \left\{ v \in \mathbb{R}^3 : v = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \right\} = \mathbb{R}^3$

$\boxed{1}$ and $\boxed{2}$ are subspaces of $\mathbb{R}^3$

Th If $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are elements of a vector space V, then $\text{Span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$ is a subspace of V.

Proof

- $\text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$ is nonempty since $\vec{0} \in \text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$
- If $\vec{v}$ is any element of $\text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$, then

$$\vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_n \vec{v}_n$$

$$\Rightarrow \beta \vec{v} = (\beta\alpha_1)\vec{v}_1 + (\beta\alpha_2)\vec{v}_2 + \dots + (\beta\alpha_n)\vec{v}_n \in \text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$$

- If $\vec{v}$ and $\vec{w}$ are arbitrary vectors in $\text{span}(\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n)$, then

$$\vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_n \vec{v}_n \quad \text{and}$$

$$\vec{w} = \beta_1 \vec{v}_1 + \beta_2 \vec{v}_2 + \dots + \beta_n \vec{v}_n .$$

$$\Rightarrow \vec{v} + \vec{w} = (\alpha_1 + \beta_1)\vec{v}_1 + (\alpha_2 + \beta_2)\vec{v}_2 + \dots + (\alpha_n + \beta_n)\vec{v}_n \in \text{span}(\vec{v}_1, \dots, \vec{v}_n)$$

Def The set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a spanning set for a vector space V iff every vector in V can be written as a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$.

Exp Which of the following are spanning set for $\mathbb{R}^3$?

1. $\left\{ e_1, e_2, e_3, \begin{pmatrix}1\\2\\3\end{pmatrix} \right\}$

- To see if a set spans $\mathbb{R}^3$, we take any vector $\begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3$ and we check if $\begin{pmatrix}a\\b\\c\end{pmatrix}$ can be written as a linear combination of $e_1, e_2, e_3, \begin{pmatrix}1\\2\\3\end{pmatrix}$.

Yes since $\begin{pmatrix}a\\b\\c\end{pmatrix} = a e_1 + b e_2 + c e_3 + 0 \begin{pmatrix}1\\2\\3\end{pmatrix}$

2. $\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix} \right\}$

- let $\begin{pmatrix}a\\b\\c\end{pmatrix} \in \mathbb{R}^3$ be arbitrary vector.
- we need to see if we can find $\alpha_1, \alpha_2, \alpha_3$ s.t

$$\begin{pmatrix}a\\b\\c\end{pmatrix} = \alpha_1 \begin{pmatrix}1\\1\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}1\\1\\0\end{pmatrix} + \alpha_3 \begin{pmatrix}1\\0\\0\end{pmatrix}$$

- This leads to the system

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= a \\ \alpha_1 + \alpha_2 &= b \\ \alpha_1 &= c \end{aligned}$$

- The coefficient matrix is nonsingular since the system $A\alpha = B$ is

$$\underbrace{\begin{bmatrix}1&1&1\\1&1&0\\1&0&0\end{bmatrix}}_{A} \begin{bmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{bmatrix} = \begin{bmatrix}a\\b\\c\end{bmatrix}$$ with $|A| = -1$

$\Rightarrow$ The system has unique solution $\alpha = \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_3\end{pmatrix} = \begin{pmatrix}c\\b-c\\a-b\end{pmatrix}$

- Thus, $\begin{pmatrix}a\\b\\c\end{pmatrix} = c\begin{pmatrix}1\\1\\1\end{pmatrix} + (b-c)\begin{pmatrix}1\\1\\0\end{pmatrix} + (a-b)\begin{pmatrix}1\\0\\0\end{pmatrix}$

- So, $\left\{ \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix} \right\}$ spans $\mathbb{R}^3$.

$\boxed{3}$ $\left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\}$

$$\alpha_1 \begin{pmatrix}1\\0\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}\alpha_1\\\alpha_2\\\alpha_1\end{pmatrix}$$

- If we take $\vec{v} = \begin{pmatrix}a\\b\\c\end{pmatrix}$ any vector in $\mathbb{R}^3$ s.t $a \neq c$, then $\vec{v}$ can not be written as a linear combination of $\begin{pmatrix}1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}0\\1\\0\end{pmatrix}$.
- For example the vector $\begin{pmatrix}2\\1\\0\end{pmatrix} \in \mathbb{R}^3$ but $\nexists$ α_1 and α_2 s.t

$$\begin{pmatrix}2\\1\\0\end{pmatrix} = \alpha_1 \begin{pmatrix}1\\0\\1\end{pmatrix} + \alpha_2 \begin{pmatrix}0\\1\\0\end{pmatrix}$$

- Thus, the set $\left\{ \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix} \right\}$ does not span $\mathbb{R}^3$.

Exp which of the following sets are spanning set for $\mathbb{R}^2$:

$\boxed{1}$ $\left\{ \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix} \right\}$

- Let $\begin{pmatrix}a\\b\end{pmatrix} \in \mathbb{R}^2$ be any vector

$$\begin{pmatrix}a\\b\end{pmatrix} = \alpha_1 \begin{pmatrix}1\\2\end{pmatrix} + \alpha_2 \begin{pmatrix}-1\\1\end{pmatrix} \Leftrightarrow \begin{matrix}\alpha_1 - \alpha_2 = a\\ 2\alpha_1 + \alpha_2 = b\end{matrix} \Leftrightarrow$$

$$\left[\begin{array}{cc|c}1 & -1 & a\\ 2 & 1 & b\end{array}\right] R_2 - 2R_1 \Leftrightarrow \left[\begin{array}{cc|c}1 & -1 & a\\ 0 & 3 & b-2a\end{array}\right] \Leftrightarrow \begin{matrix}\alpha_1 = \frac{a+b}{3}\\ \alpha_2 = \frac{b-2a}{3}\end{matrix}$$

- Hence, $\begin{pmatrix}a\\b\end{pmatrix} = \left(\frac{a+b}{3}\right)\begin{pmatrix}1\\2\end{pmatrix} + \left(\frac{b-2a}{3}\right)\begin{pmatrix}-1\\1\end{pmatrix}$

$\Rightarrow \left\{ \begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix} \right\}$ spans $\mathbb{R}^2$

$\boxed{2}$ $\left\{ \begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}1\\-2\end{pmatrix}, \begin{pmatrix}2\\-4\end{pmatrix} \right\}$ is not a spanning set for $\mathbb{R}^2$ since $v = \begin{pmatrix}1\\1\end{pmatrix} \in \mathbb{R}^2$ but v can not be written as a linear combination of $\begin{pmatrix}-1\\2\end{pmatrix}, \begin{pmatrix}1\\-2\end{pmatrix}, \begin{pmatrix}2\\-4\end{pmatrix}$. That is

$$* \ldots \quad \begin{pmatrix}1\\1\end{pmatrix} = \alpha_1 \begin{pmatrix}-1\\2\end{pmatrix} + \alpha_2 \begin{pmatrix}1\\-2\end{pmatrix} + \alpha_3 \begin{pmatrix}2\\-4\end{pmatrix}$$ implies that the system

$$\left[\begin{array}{ccc|c}-1 & 1 & 2 & 1\\ 2 & -2 & -4 & 1\end{array}\right] R_2 + 2R_1 \Rightarrow \left[\begin{array}{ccc|c}-1 & 1 & 2 & 1\\ 0 & 0 & 0 & 3\end{array}\right]$$ is inconsistent.

So $\nexists$ α_1, α_2 s.t $*$ holds.