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Control Systems
KOM3711
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↳ Teaching
↳ Control Systems

2 midterm exams	2 x 30%
1 Final exam	1 x 40%

1. Introduction
2. Math. preliminaries
3. Modeling
4. First order syst.
5. 2nd order syst.
6. Steady-state errors
7. Stability
8. Design

Introduction to Control

Loosely speaking, control is the process of getting "something" to do what you want it to do.

↳ can be almost anything: Ex: aircrafts, spacecrafts, cars, machines, robots, radars, telescopes, etc.

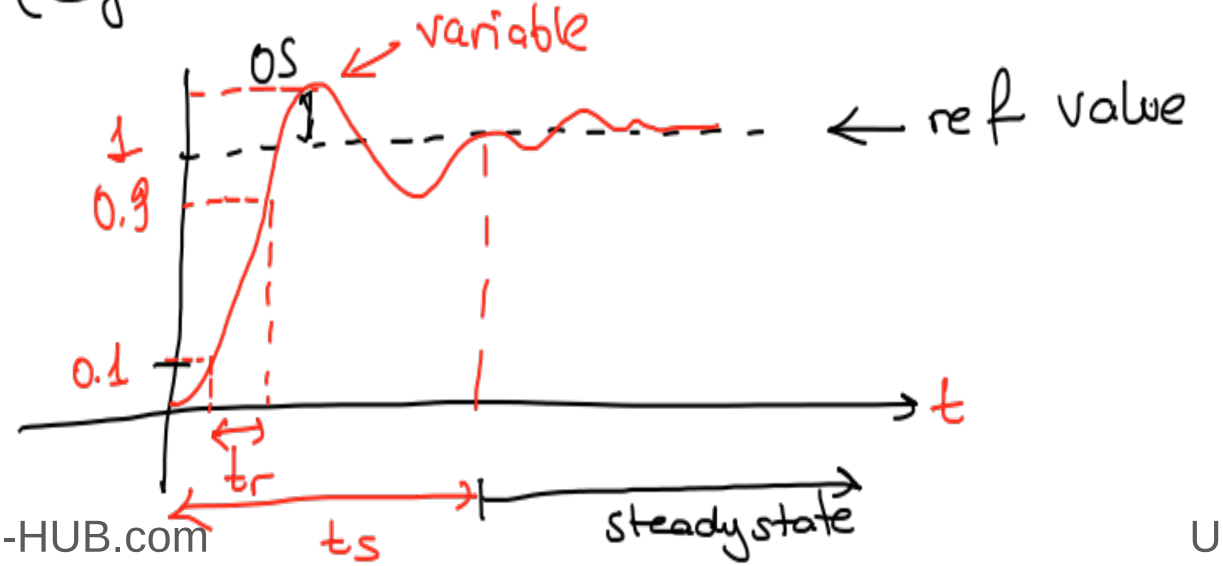
Some less obvious examples: energy systems, the economy, biological systems, human body.

Manual Control: human-machine interaction. Driving car, bike, etc.

Automatic Control: No human exists.

Definition: Control is the process of causing a system variable to conform to some desired value which is called "reference" value.

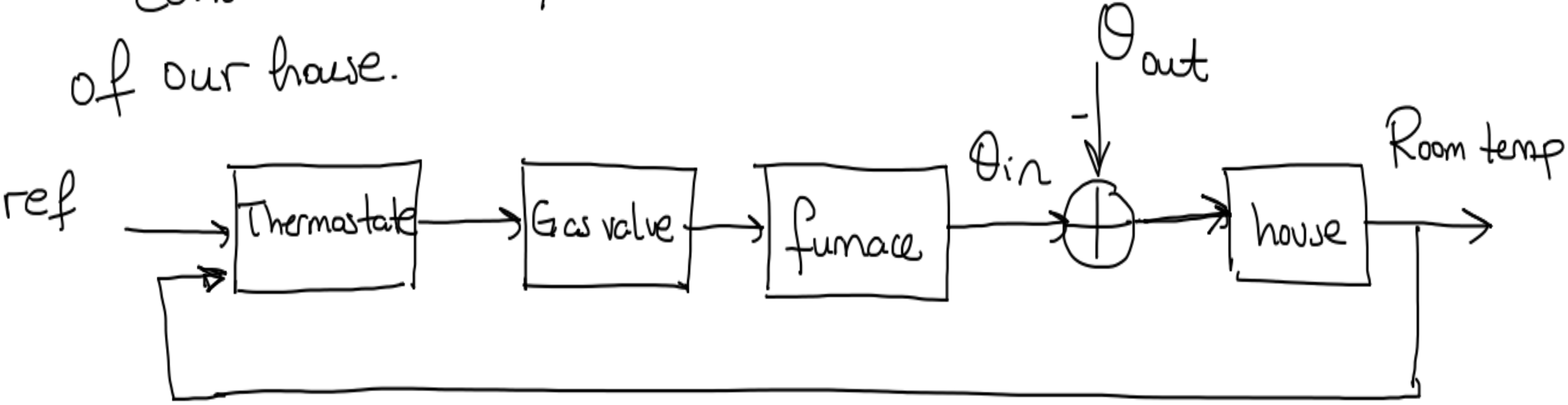
(eg. variable = temp in a climate control system)



Definition: Feedback is the process of measuring the controlled variable (e.g. temp) and using that information to influence the value of the controlled variable.

An illustrative example of a fb system

Consider the system designed to maintain the temperature of our house.



- Central component = process or plant or system one of whose variable we want to control

  e.g. Plant = house

  Variable = Room Temp

- Disturbance : some system input that we do not control.

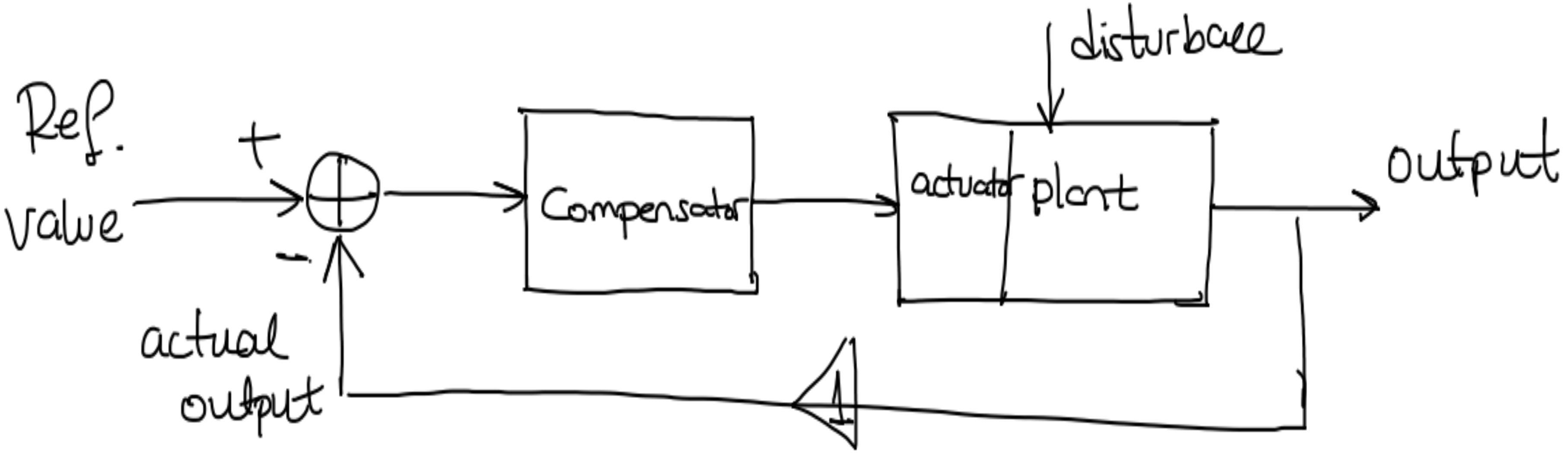
  e.g. disturbance = θ_{out} temp. loss.

- Actuator : device that influences controlled variable

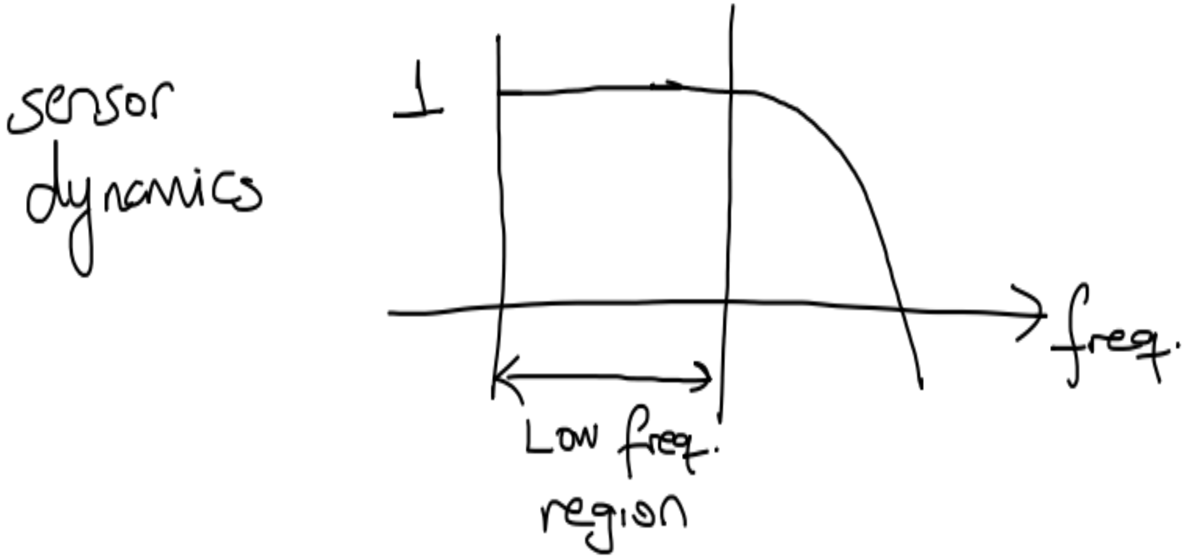
  e.g : actuator = furnace + gas value.

- Reference sensor: measures the desired system output
- output sensor: " " actual " "
- Controller / Compensator : device that computes the control effort

An even more abstract block diagram



We assumed that we have a perfect sensor.



The Control Problem Methodology

The objectives of any control system design:

1. "Reject" disturbances (plant response to unwanted input
2. Acceptable steady-state errors
3. Acceptable transient-response
4. Minimize the sensitivity to plant parameter changes (robust control)

Solutions are reached via the methodology:

iterations

1. Choose an acceptable output sensor.
2. Choose an appropriate actuator
3. Develop plant, actuator, sensor equations (models)
4. Design the controller based on the model
5. Evaluate the design, analytically, simulations, prototype

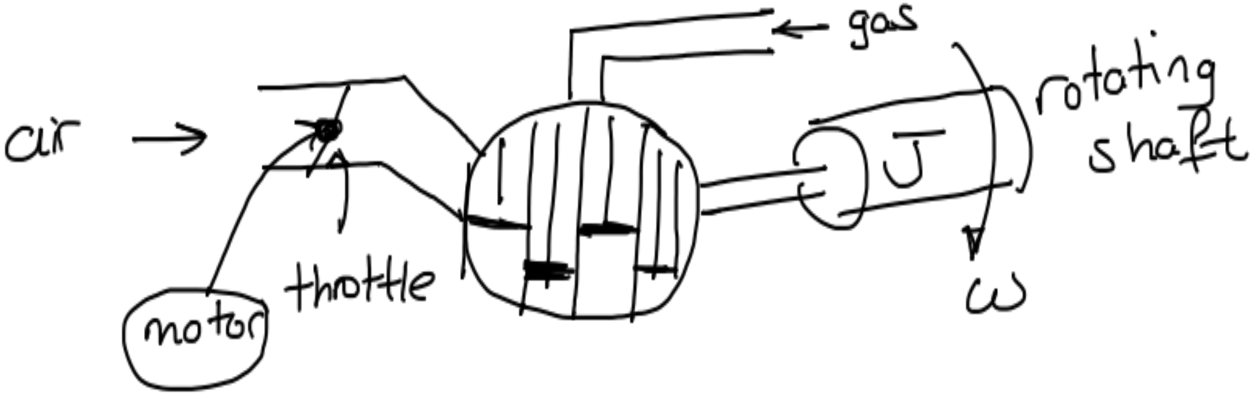
A First Time Analysis

- Suppose, we wish to design a cruise-control system for an automobile.

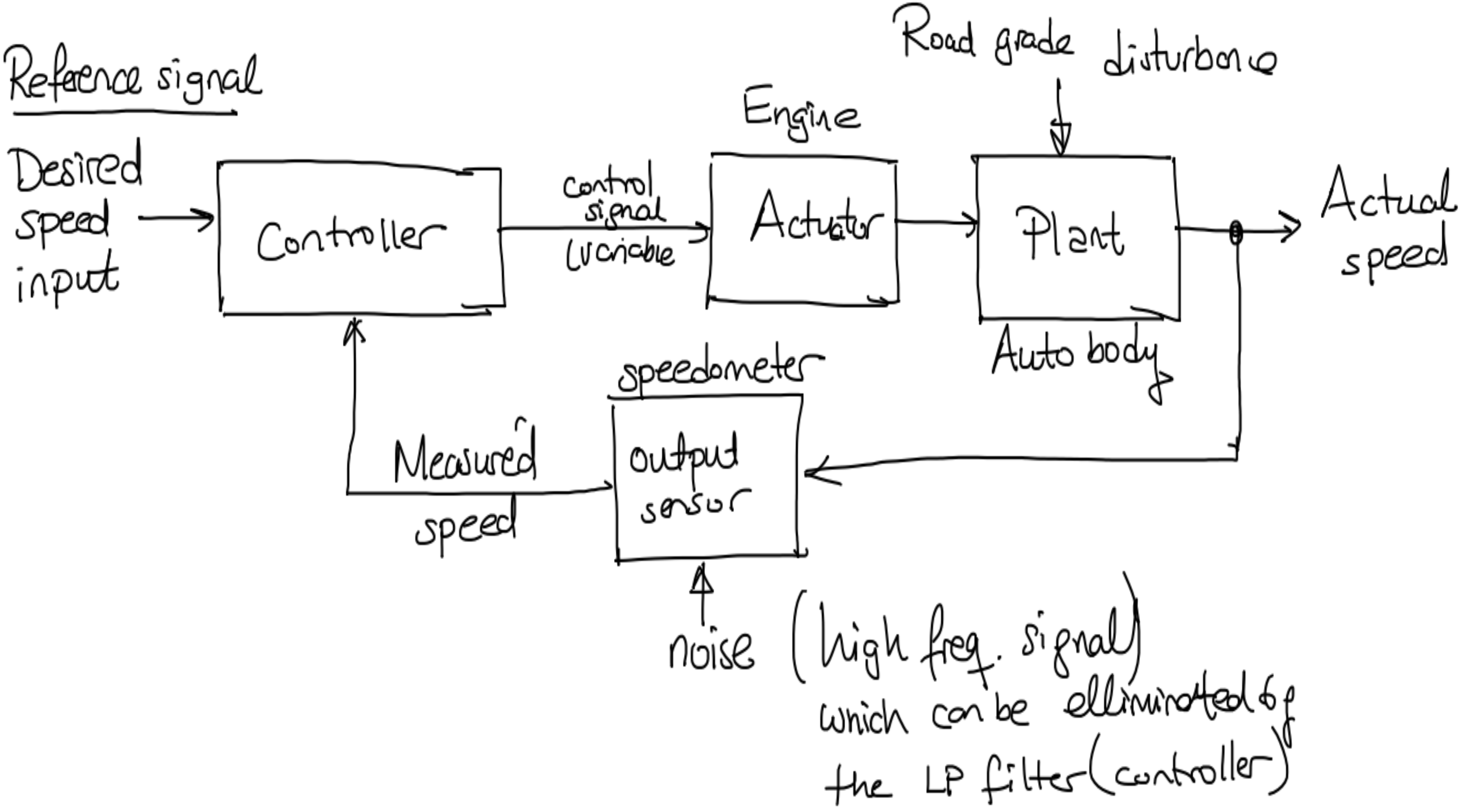
- Generally, we wish to control the cart speed.

Following our design methodology:

1. output sensor = speedometer
2. actuator = throttle and engine



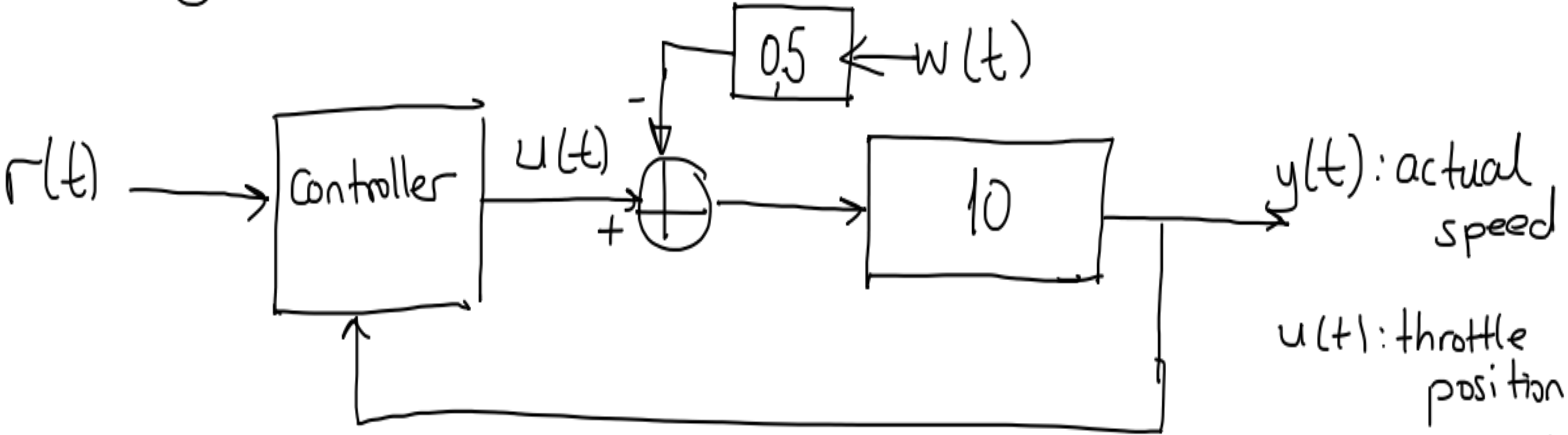
Reference signal



(high freq. signal) which can be elliminated by the LP filter (controller)

3. Model of the system:

Assumptions
- operating speed ≈ 55 km/h
- 1% change in throttle ⟹ 10km/h change in speed
- 1% change in road grade ⟹ 5km/h " " "
- speedometer accurate to a fraction of 1km/h



u(t): throttle position(%)

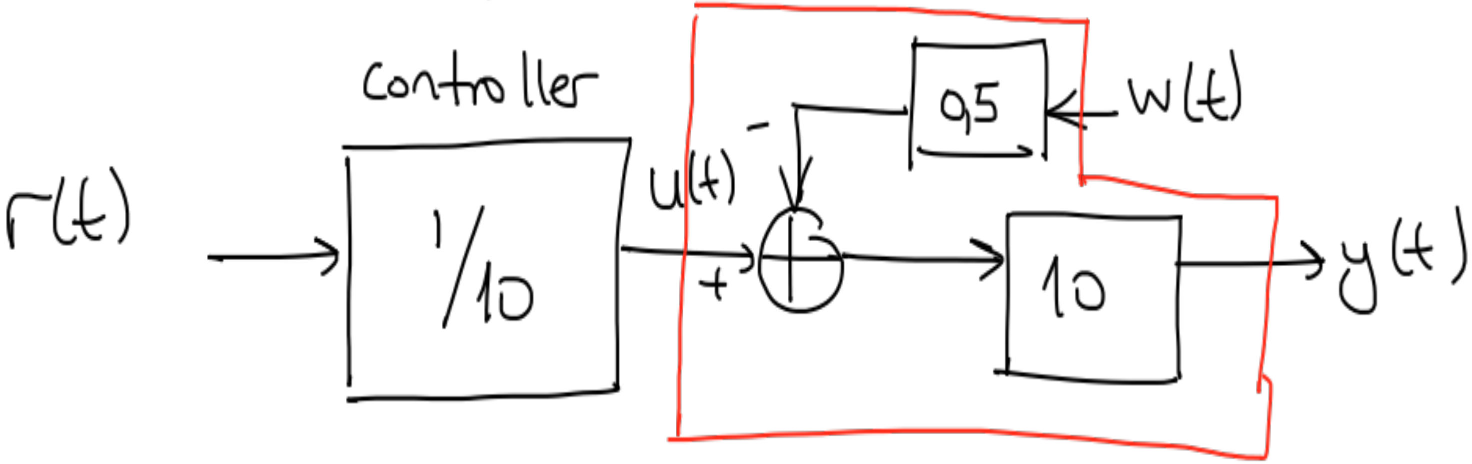
w(t): road grade%

r(t): reference speed

4. Design the controller:

Compensator: $\left(\frac{1}{10}\right)$

First attempt = " open-loop case "



$$y_{ol}(t) = 10\left(u(t) - 0.5w(t)\right) = 10\left(\frac{1}{10}r(t) - 0{,}5w(t)\right)$$

$$u(t) = \frac{1}{10}r(t)$$

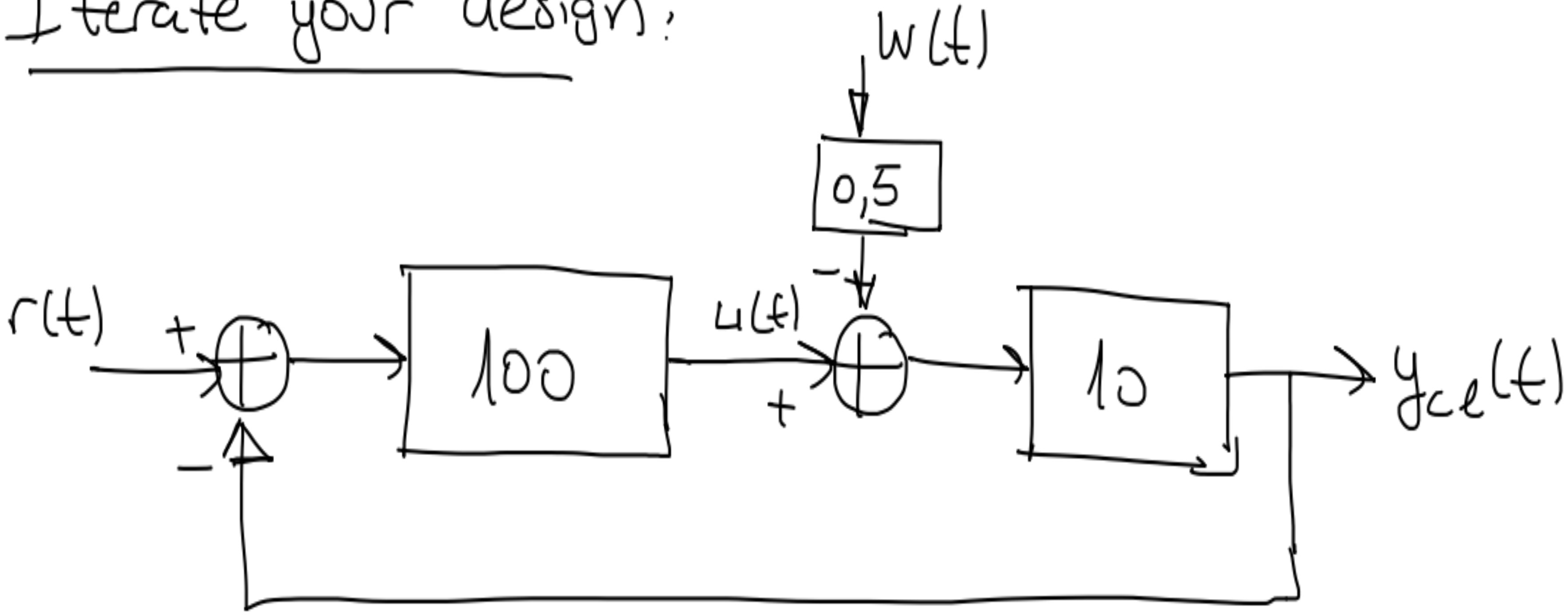
$$\Rightarrow y_{ol}(t) = r(t) - 5w(t)$$

5. Evaluate the design:

$$\text{Performance Measure} = 100\left(\frac{r(t) - y_{ol}(t)}{r(t)}\right) \swarrow \text{percentage error}$$

			Performance M.
• $r(t) = 55\,km/h$	, $w(t) = 0$	$y_{ol}(t) = 55\,km/h$	No error
• $r(t) = 55\,km/h$	, $w(t) = 1$	$y_{ol}(t) = 50\,km/h$	$\approx$ 10% error
• $r(t) = 55\,km/h$	, $w(t) = 2$	$y_{ol}(t) = 45\,km/h$	$\approx$ 20% error
• $r(t) = 55\,km/h$	, $w(t) = 0$	, (but assume that your model is not correct and your gain is 9)	$y_{ol}(t) = 49.5\,km/h$ $\approx$ 10% error

Iterate your design:



$$y_{cl}(t) = 10u(t) - 5w(t)$$

$$u(t) = 100\left(r(t) - y_{cl}(t)\right)$$

$$y_{cl}(t) = 10\left(100\left(r(t) - y_{cl}(t)\right)\right) - 5w(t)$$

$$y_{cl}(t) = 0.999\, r(t) - 0.005\, w(t)$$

5. Evaluate the design:

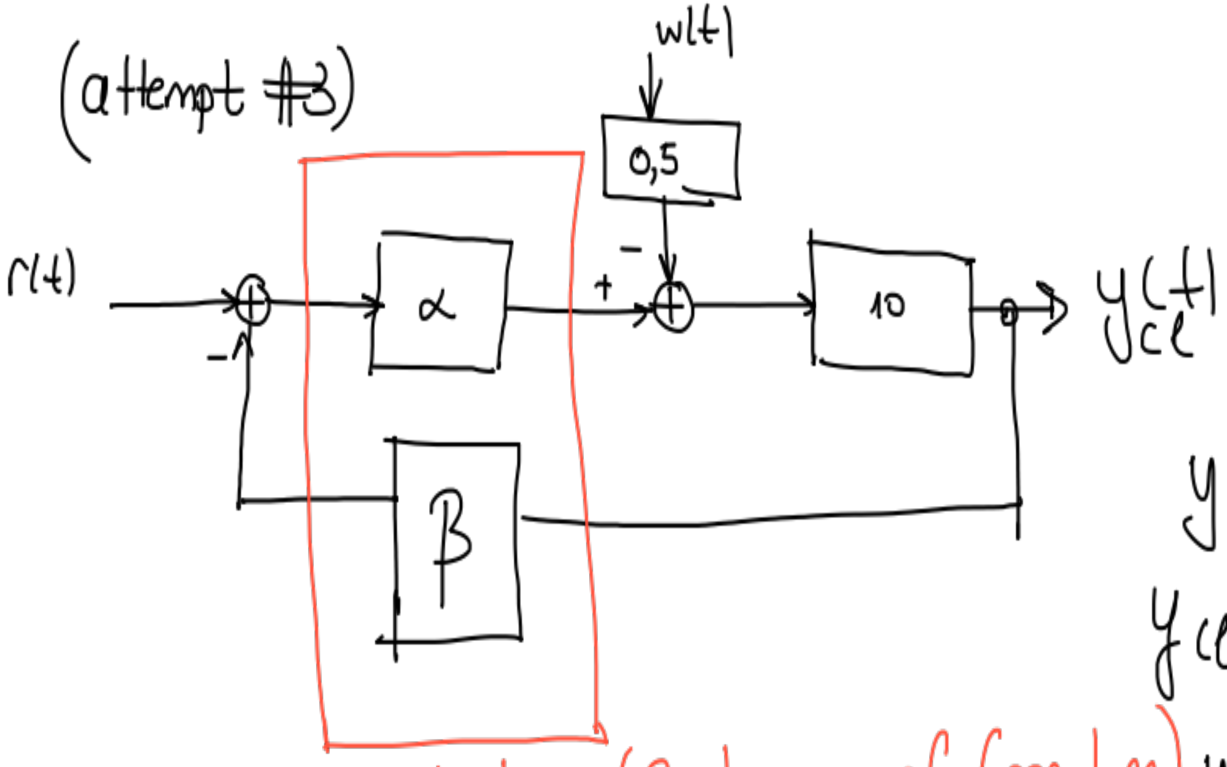
- $r(t) = 55$ km/h , $w(t) = 0$, $y_{cl}(t) = 54{,}945$
- $r(t) = 55$ km/h , $w(t) = 1$, $y_{cl}(t) = 54{,}94$
- $r(t) = 55$ km/h , , $w(t) = 2$, $y_{cl}(t) = 54{,}935$
- $r(t) = 55$ km/h , , $W(t) = 10$, $y_{cl}(t) = 54{,}895$

$\cong 0{,}2\,\%$ error

Punchline: Feedback syst. rejects disturbances

Feedback syst. has steady-state errors.

(attempt #3)



controller (2 degree-of-freedom)

$$y_{cl}(t) = 10\left(\alpha r(t) - \beta\alpha y_{cl}(t)\right) - 5w(t)$$

$$y_{cl}(t) = 10\alpha r(t) - \alpha 10\beta y_{cl}(t) - 5w(t)$$

$$y_{cl}(t) = \frac{10\alpha}{\underbrace{1+10\alpha\beta}_{}} r(t) - \frac{5}{1+10\alpha\beta} w(t)$$

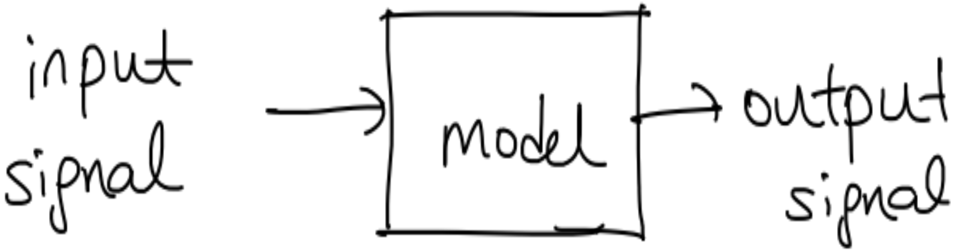
set to be $\beta = 1 - \frac{1}{10\alpha}$

when $r(t) = 55 km/h$, $w(t) = 0 \Rightarrow e_{ss} = 0$

steady-state error

System Modeling in Time-domain

- We use the term model to refer to a set of mathematical equations used to represent a physical system. These equations relate system's output signal to its input.



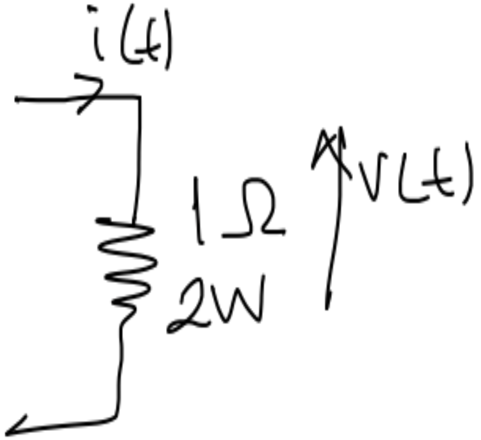
A model is required in order to

1. Understand the system behavior
2. Design the controller

There are 2 approaches to model:

1. Analytical modeling - we'll focus on this method.
2. Emprical system identification

EX: Consider 1Ω, 2W resistor



- Ohm's law $\quad v(t) = i(t) R$
- Apply 1V. What happens?

→ 1A of current is predicted to flow

$$\text{Power dissipated} = \frac{V^2}{R} = 1W$$

Apply 10V

$\rightarrow$ 10 A of current is predicted

Dissipated power = $V^2/R = 100\,W$

$\Rightarrow$ model will no longer be accurate

$\Rightarrow$ True model depends on input signal.

$\Rightarrow$ Model is accurate only in a certain range of input signals.

LTI System (Linear Time Invariant) System)

Time-Invariance: A system is either time varying or time-invariant, not-both

A time-invariant system does not change its fundamental behaviour over different periods of time.

It's parameter values are constant

A TI system satisfies the following property:

$x(t-\tau) \longrightarrow y(t-\tau)$ when $x(t) \longrightarrow y(t)$ $(\forall t, \forall \tau)$

Test method:

- Input $x_1(t)$ is applied to the system and the output $y_1(t)$ is measured
- Input $x_2(t) = x_1(t-\tau)$ is applied to the system, the measured output $y_2(t)$ is recorded
- If $y_2(t) = y_1(t-\tau)$ for all possible delays $(\tau > 0)$ and signals $x_1(t)$, then we conclude that the system is TI.

EX: Consider the system $y(t) = (x(t))^2$

TI / TV ?

$x_1 \longmapsto y_1$

$y_1(t) = (x_1(t))^2$

apply $x_2(t) := x_1(t-\tau)$ so $y_2(t) = (x_1(t-\tau))^2$

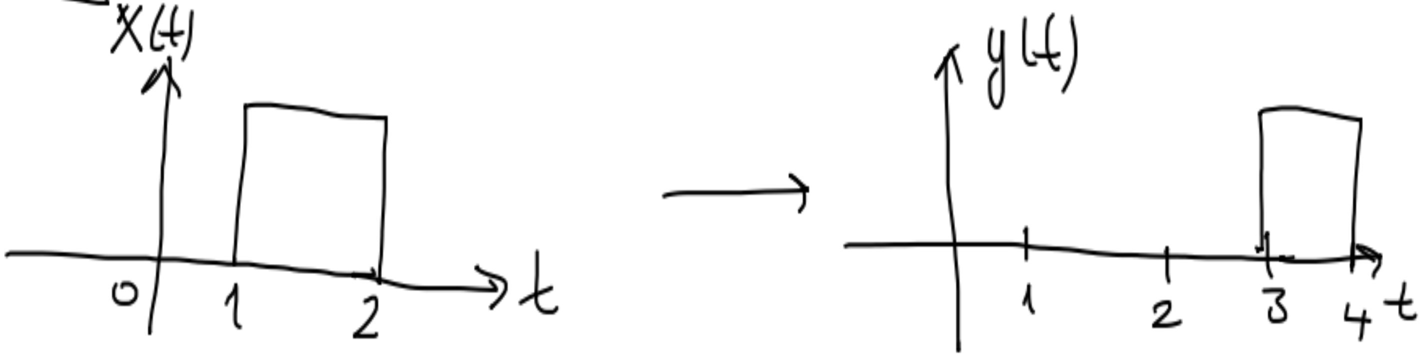
$y_1(t-\tau) = (x_1(t-\tau))^2$

identical

$\therefore$ square-law system is TI

Ex: (delay operator) TI/TV ?

$$x(t) \longrightarrow \boxed{\lambda} \longrightarrow y(t) = x(t-\lambda), \quad \lambda \geq 0$$



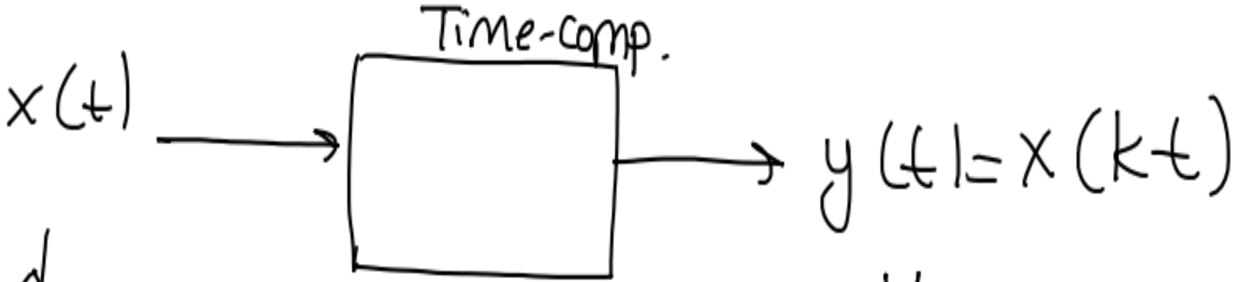
$$\longrightarrow y_1(t-\tau) = x_1(t-\tau-\lambda) \quad \lambda = 2$$

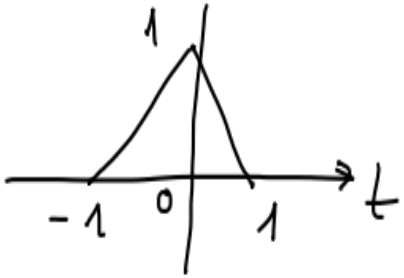
$$x_1(t) \longmapsto y_1(t) = x_1(t-\lambda)$$

$$x_2(t) := x_1(t-\tau) \longmapsto y_2(t) = x_2(t-\lambda) = x_1(t-\lambda-\tau)$$

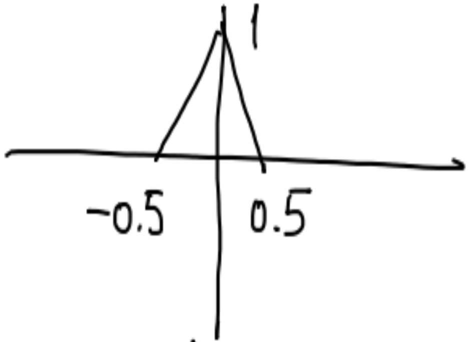
$$= y_1(t-\tau)$$

EX: "time-compressor" (TI / TV ?)





k = 2



Test: $x_1(t) \longmapsto y_1(t) = x_1(kt) \longrightarrow y_1(t-\tau) = x_1(k(t-\tau))$

$$= x_1(kt - k\tau)$$

$$x_2(t) \longmapsto y_2(t) = x_2(kt) \qquad \neq ?$$

$$x_2(t) := x_1(t-\tau) \longrightarrow y_2(t) = x_2(kt) = x_1(kt-\tau)$$

(Answer = TV)

Linearity: For Linear Systems, if $x_1(t) \longmapsto y_1(t)$ and

$$x_2(t) \longmapsto y_2(t)$$

then $x_3(t) = \alpha x_1(t) + \beta x_2(t) \longmapsto y_3(t) = \alpha y_1(t) + \beta y_2(t)$

for any $\alpha, \beta, x_1(t), x_2(t)$

To test linearity, we must:

- input $x_1(t)$ to the system, measure $y_1(t)$
- input $x_2(t)$ " " " , " $y_2(t)$
- input $x_3(t) = \alpha x_1(t) + \beta x_2(t)$ to the syst, measure $y_3(t)$
- if $y_3(t) = \alpha y_1(t) + \beta y_2(t)$ $\forall \alpha, \forall x_1, \forall x_2, \forall \beta \Rightarrow$ syst.(L)

EX: is the following system, described by diff eq.

$\dot{y}(t) + t y(t) = x(t)$ linear or not?

$$x(t) \longrightarrow \boxed{\dot{y}(t)+ty(t)=x(t)} \longrightarrow y(t)$$

$x_1(t) \longrightarrow y_1(t)$: $\dot{y}_1(t) + t y_1(t) = x_1(t)$

$x_2(t) \longrightarrow y_2(t)$: $\dot{y}_2(t) + t y_2(t) = x_2(t)$

$x_3(t) = \alpha x_1(t) + \beta x_2(t) \longmapsto y_3(t) = ?$

$y_3(t) = \dot{y}_3(t) + t y_3(t) = \alpha x_1(t) + \beta x_2(t)$

$$\dot{y}_3(t) + t\,y_3(t) = \alpha\left(\dot{y}_1(t) + t\,y_1(t)\right) + \beta\left(\dot{y}_2(t) + t\,y_2(t)\right)$$

$$\dot{y}_3(t) + t\,y_3(t) = \frac{d}{dt}\left(\overbrace{\alpha y_1(t) + \beta y_2(t)}^{y_3(t)}\right) + t\left(\underbrace{\alpha y_1(t) + \beta y_2(t)}_{y_3(t)}\right)$$

$$\overbrace{\dot{y}_3(t) + t\,y_3(t)} = \underbrace{\dot{y}_3(t) + t\,y_3(t)} \Rightarrow \text{LINEAR (L)}$$

Ex: (Square-Law Syst) $y(t) = (x(t))^2$

Test: $x_1(t) \longmapsto y_1(t) = (x_1(t))^2$

$x_2(t) \longmapsto y_2(t) = (x_2(t))^2$

$$x_3(t) = \alpha x_1 + \beta x_2 \longmapsto y_3 = (\alpha x_1 + \beta x_2)^2 = \alpha^2 x_1^2 + \beta^2 x_2^2 + 2\alpha\beta x_1 x_2$$

$$\overset{?}{=} \alpha y_1 + \beta y_2 = \alpha x_1^2 + \beta x_2^2$$

$\Rightarrow$ Square-law system is not Linear

Modeling of Physical Systems

Dynamics of mechanical systems (translational motion)

Translational motion: Newton's 2nd Law

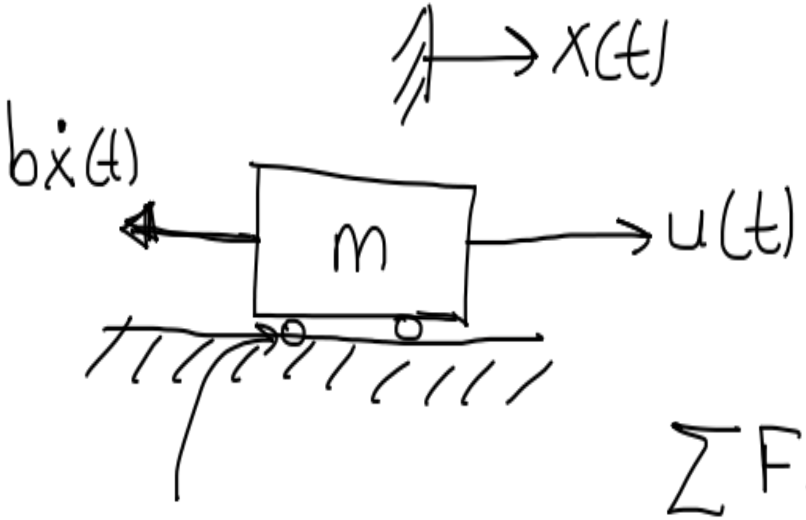
$$\sum F = m \cdot a$$

$$\underset{[N]}{\sum F} = \underset{[kg]}{m} \cdot \underset{[m/s^2]}{a}$$

That is, the vector sum of forces = mass of the obj x acceleration

EX: (cruise control model)

1. Assume rotational inertia is negligible
2. Assume that the friction is proportional to car's speed.



b : viscous friction constant

$$\sum F = m \cdot a$$

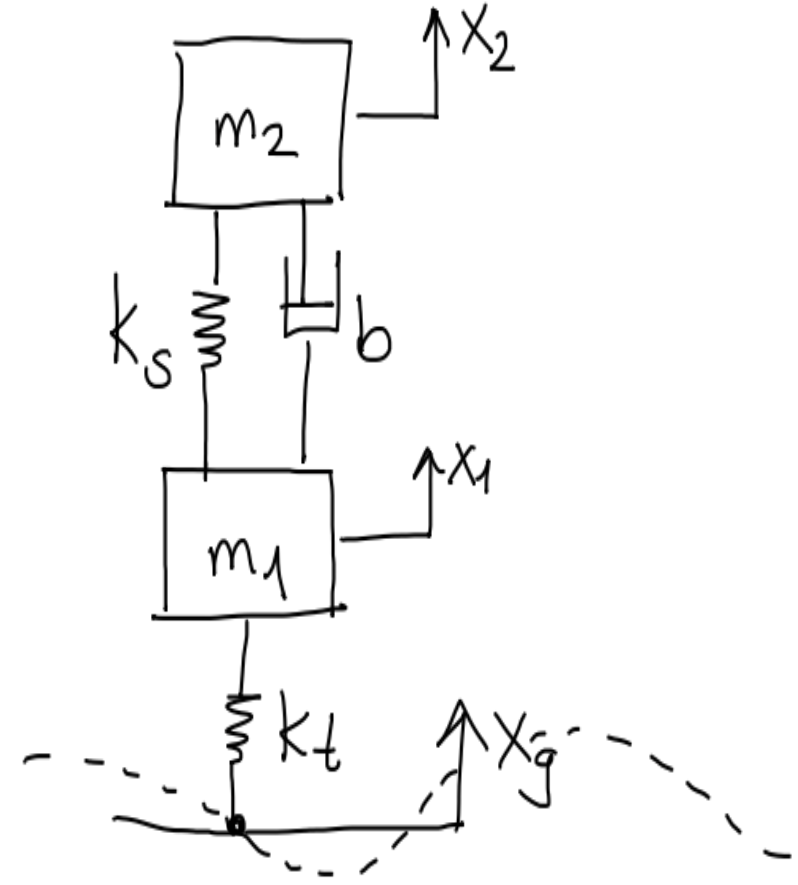
$$u(t) - b\dot{x}(t) = m\ddot{x}(t)$$

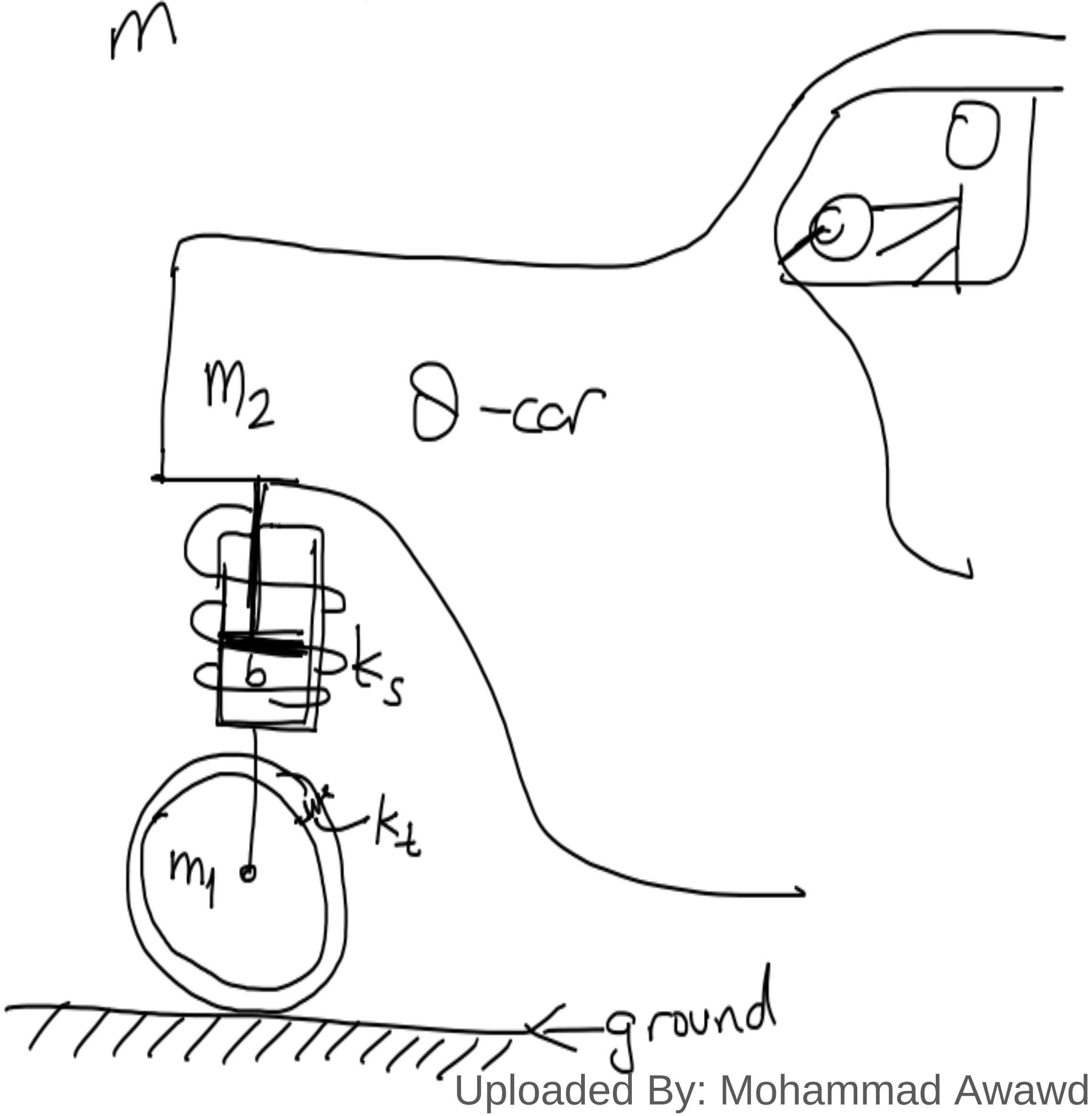
$$\Rightarrow \boxed{\ddot{x}(t) + \frac{b}{m}\dot{x}(t) = \frac{1}{m}u(t)}$$ 2nd order ODE

If the variable of interest is the speed ($v(t) = \dot{x}(t)$)

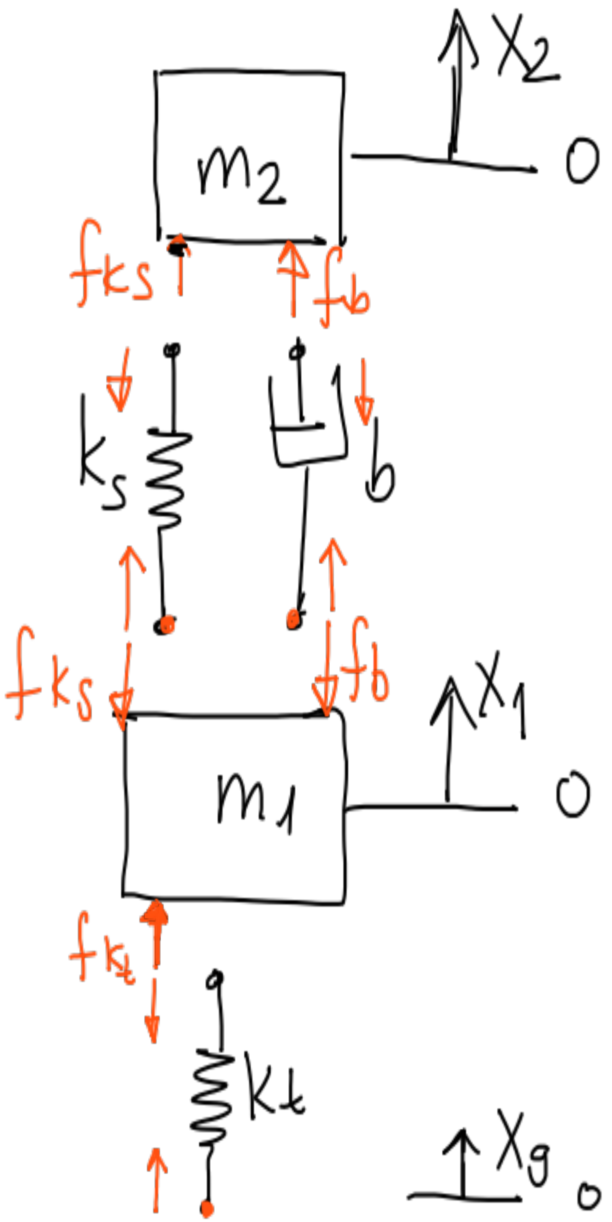
$$\Rightarrow \dot{v}(t) + \frac{b}{m} v(t) = \frac{1}{m} u(t)$$

EX: (car suspension)





free-body diagram:



① $f_{kt} - f_{ks} - f_b = m_1 \ddot{x}_1$

② $f_{ks} + f_b = m_2 \ddot{x}_2$

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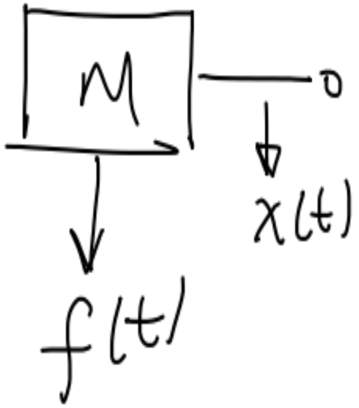
① $k_t \cdot (x_g - x_1) - k_s(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2) = m_1 \ddot{x}_1$

② $k_s(x_1 - x_2) + b(\dot{x}_1 - \dot{x}_2) = m_2 \ddot{x}_2$

Translational motion

MASS



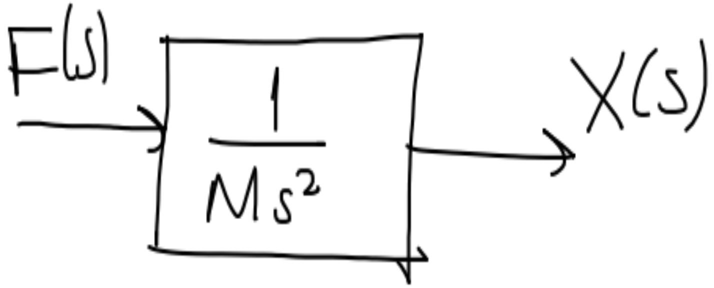
$X(0)=0\ ,\ \dot{x}(0)=0$

$f(t)=M\ddot{x}(t)$

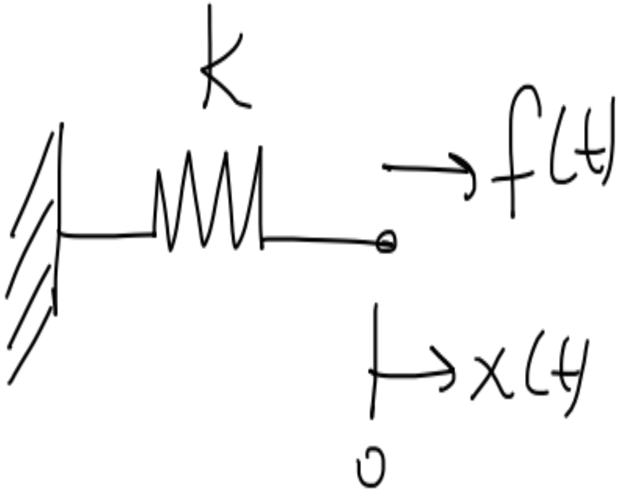
m: mass (kg)

f(t): force (N)

$F(s)=Ms^2X(s)$



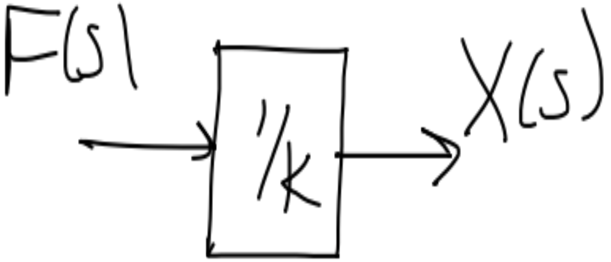
spring



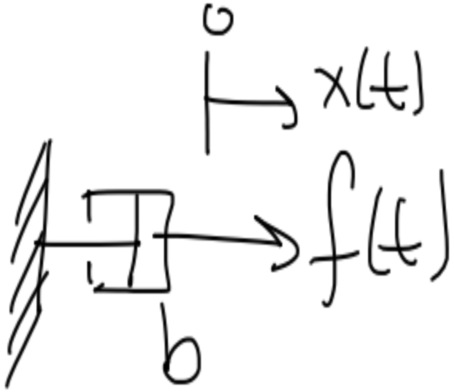
$$f(t) = k\,x(t)$$, $x(0)=\dot{x}(0)=0$

$$F(s) = k\,X(s)$$

k = spring const.
(N/m)



Damper



$x(0)=\dot{x}(0)=0$

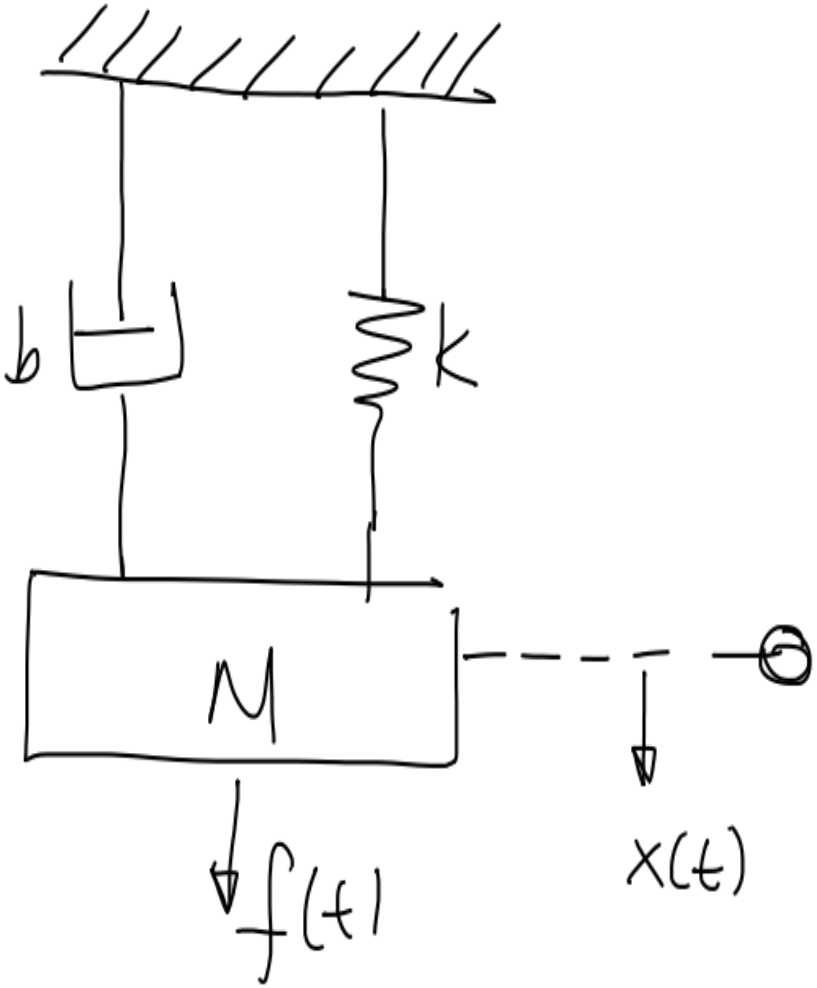
$$f(t) = b\,\dot{x}(t)$$

$[N]$ $\left[m/s\right]$

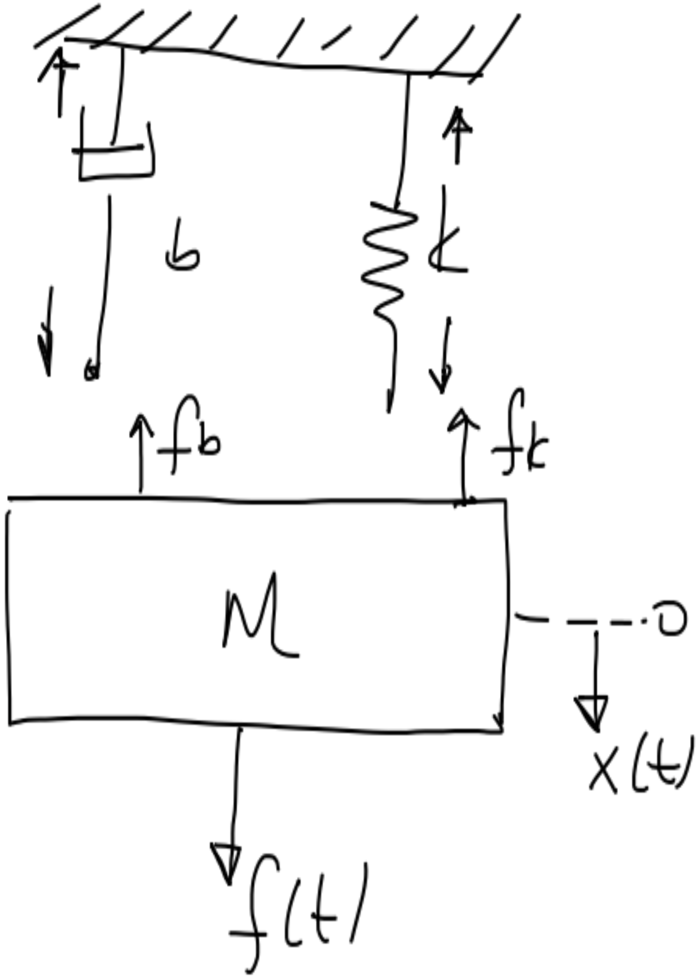
$$\downarrow$$

$$\left[N/m/s = Ns/m\right]$$

EX:



free-body diagram



$\Rightarrow \quad f - f_b - f_k = M\ddot{x} \qquad x(0) = \dot{x}(0) = 0$

$\Rightarrow \quad f - b\dot{x} - kx = M\ddot{x} \Rightarrow F(s) - bsX(s) - kX(s) = Ms^2X(s)$

$$F(s) \longrightarrow \boxed{\dfrac{1}{Ms^2+bs+k}} \longrightarrow X(s)$$

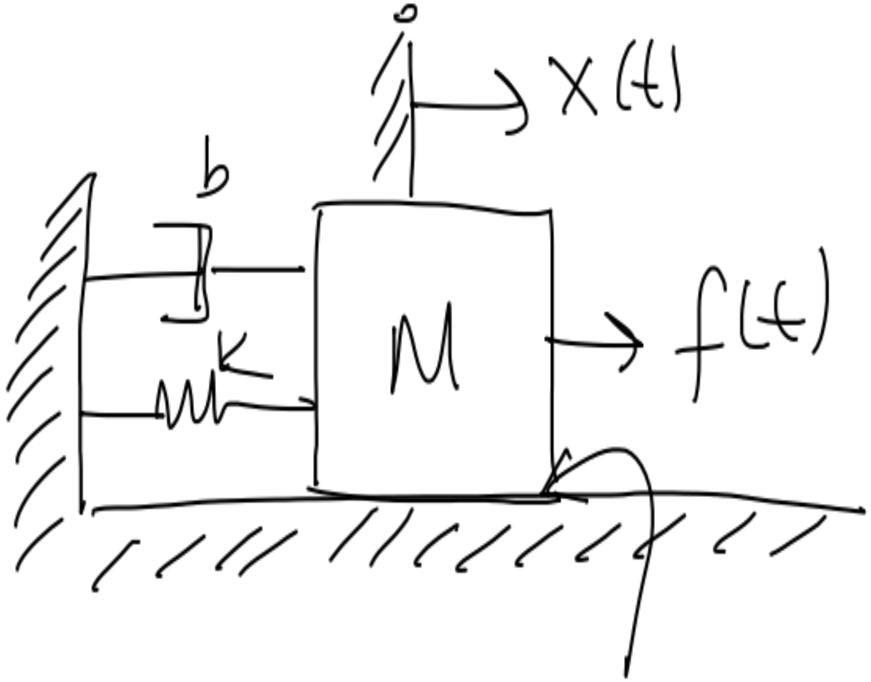
2nd order ODE

$$s^2 M X(s) = F(s) - k X(s) - b s X(s)$$

$$X(s)\left[s^2 M + bs + k\right] = F(s)$$

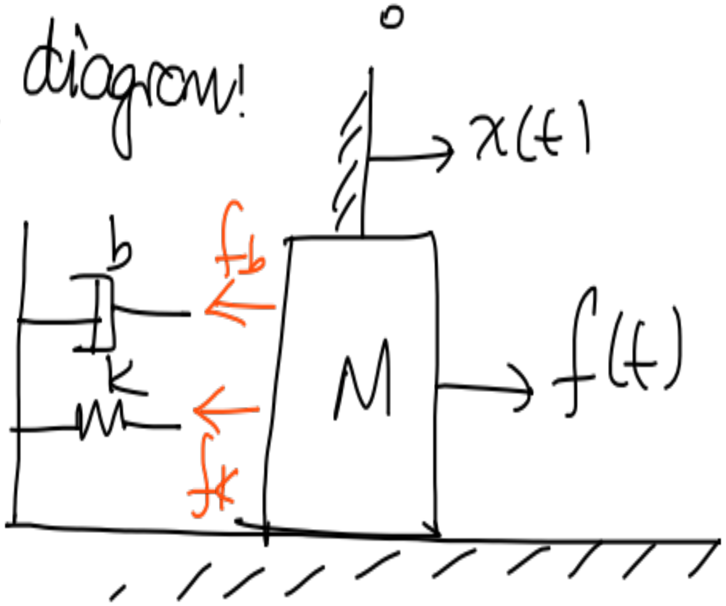
$$X(s) = \frac{1}{Ms^2+bs+k} F(s)$$

Ex:



b_v : viscous friction const. (proportional to speed)

free-body diagram:



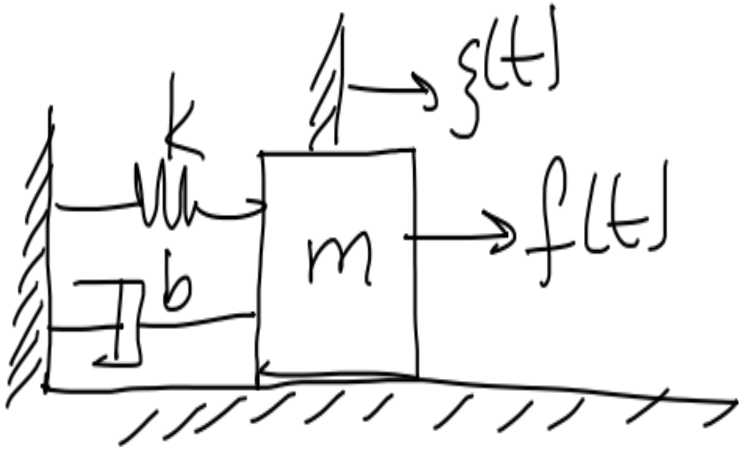
$$f - f_b - f_k - f_v = M\ddot{x}$$

$$f - b\dot{x} - kx - b_v . \dot{x} = M\ddot{x}$$

$$f - \dot{x}(b + b_v) - kx = M\ddot{x}$$

$$F(s) - (b+b_v)sX(s) - kX(s) = Ms^2X(s)$$

State-Space Models of Translational mechanical Systems



The dynamic of motion

$$m\ddot{\xi} = f - k\xi - b\dot{\xi}$$

$$\Rightarrow m\dot{x}_2 = u - k\,x_1 - b\,x_2$$

$$\xi =: x_1$$

$$\dot{\xi} =: x_2 = \dot{x}_1$$

$$\ddot{\xi} = \dot{x}_2$$

$$f =: u$$

$$\vec{x} \triangleq \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{m}x_1 - \frac{b}{m}x_2 + \frac{1}{m}u$$

$$\vec{x} = \begin{bmatrix} x_1 \\ \\ x_2 \end{bmatrix} \begin{matrix} \leftarrow \text{state} \\ \\ \leftarrow \text{state} \end{matrix}$$

$$\frac{d}{dt}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} \underset{\vec{u}}{\underset{\downarrow}{u}}$$

$\vec{x}$: state vector

$$\boxed{\dot{\vec{x}} = A\vec{x} + B\vec{u}}$$

state-space form

State-space equation

Let's assume we're interested in the output velocity

$$\Rightarrow \quad y(t) = \dot{\xi}(t) \qquad \vec{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \xi(t) \\ \dot{\xi}(t) \end{bmatrix}$$

$$\dot{\xi}(t) = f(\vec{x})$$

$$y(t) = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C}{}_{1\times 2} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \end{bmatrix}}_{D} u$$

In general any LTI system can be described by

$$\dot{x} = \overset{[n\times n]}{A}x + \overset{[n\times m]}{B}u \leftarrow \text{state equation}$$

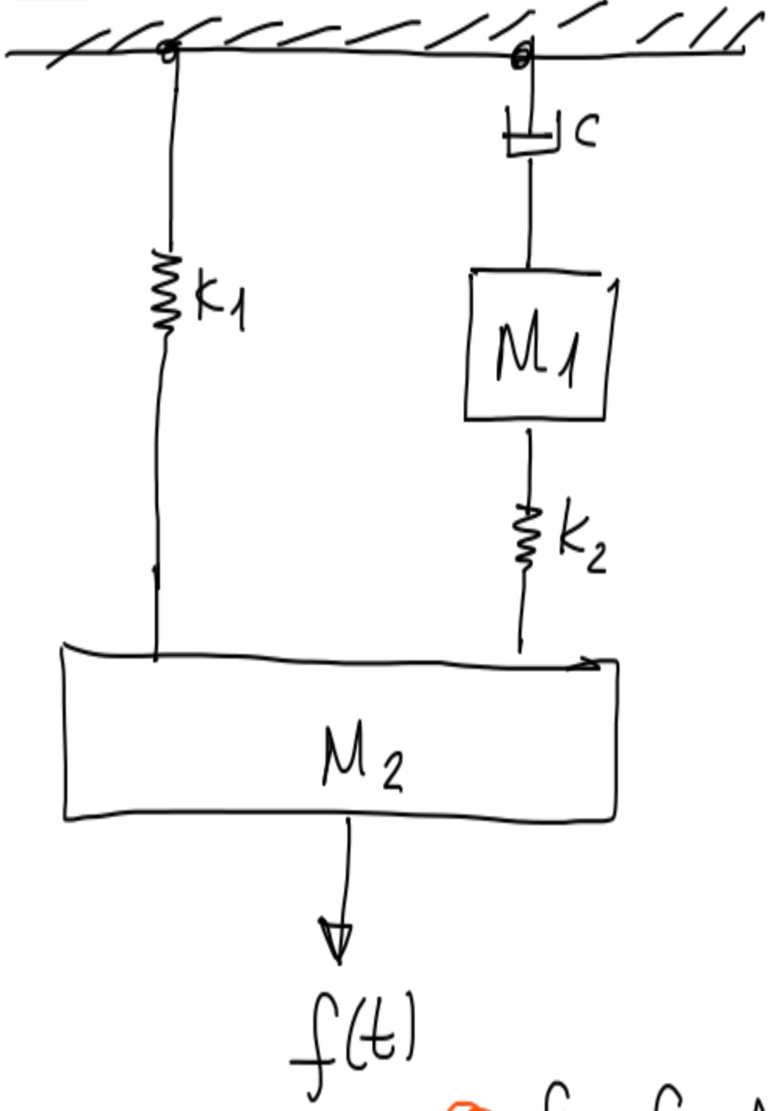
$$y = \underset{[p\times n]}{C}x + \underset{[p\times m]}{D}u \leftarrow \text{output eqn.}$$

where $\underset{\substack{\uparrow \\ \text{state vector}}}{x} \in \mathbb{R}^n = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

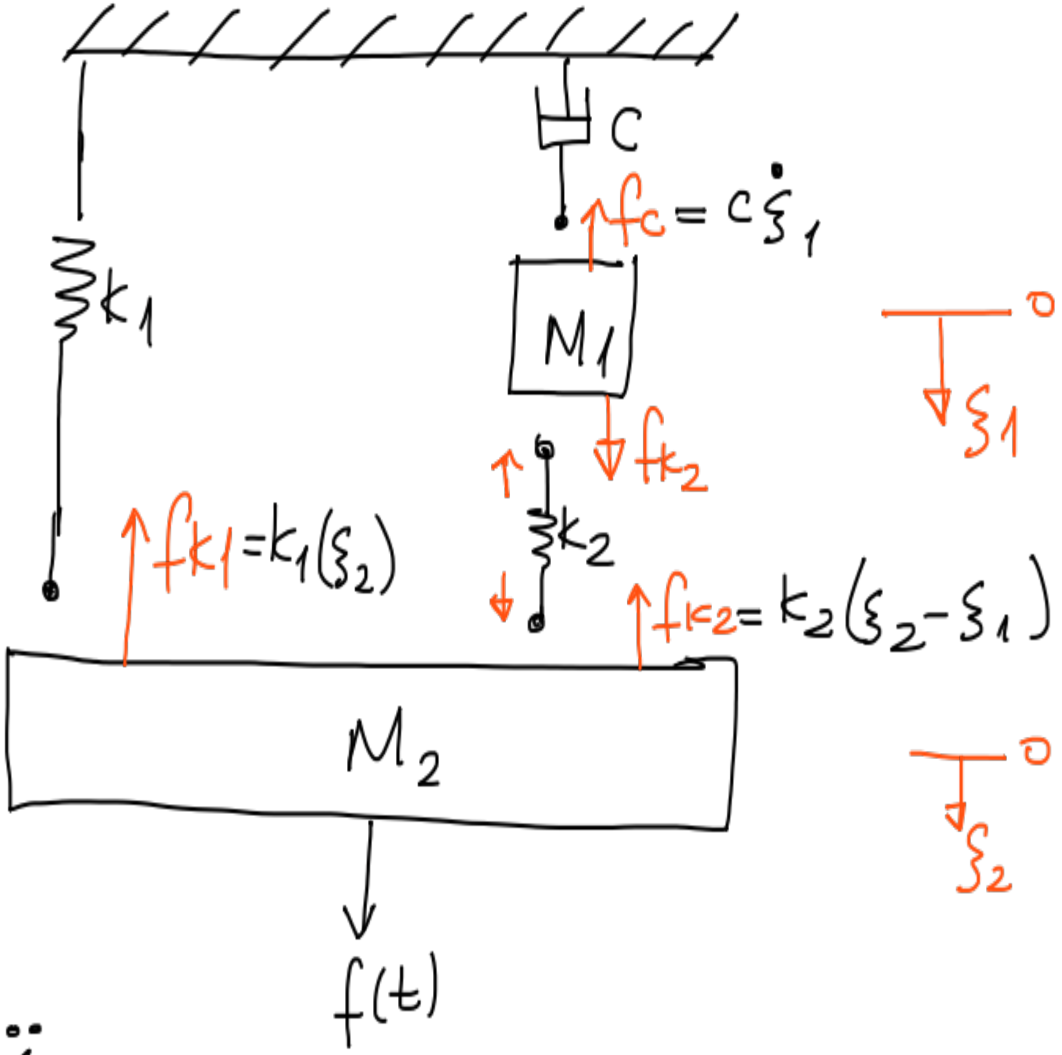
$u \in \mathbb{R}^m = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$ is the input vector

$y(t) \in \mathbb{R}^p = \begin{bmatrix} y_1 \\ \vdots \\ y_p \end{bmatrix}$ is the output vector of the system

EX: obtain SS equation for the system given below:



free-body diagram:



① $f_{k_2} - f_c = M_1\ddot{\xi}_1$

② $f - f_{k_1} - f_{k_2} = M_2\ddot{\xi}_2$

① $f_{k_2} - f_c = M_1\ddot{\xi}_1 \quad \Rightarrow \quad k_2(\xi_2-\xi_1) - c\dot{\xi}_1 = M_1\ddot{\xi}_1$

② $f - f_{k_1} - f_{k_2} = M_2\ddot{\xi}_2 \quad \Rightarrow \quad f - k_1\xi_2 - k_2(\xi_2-\xi_1) = M_2\ddot{\xi}_2$

$$\Rightarrow \vec{x} \triangleq \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}\xi_1\\\xi_2\\\dot{\xi}_1\\\dot{\xi}_2\end{bmatrix}, \quad u \triangleq f \quad \Rightarrow \left.\begin{array}{l} k_2(x_2-x_1) - cx_3 = M_1\dot{x}_3 \\ u - k_1x_2 - k_2(x_2-x_1) = M_2\dot{x}_4 \end{array}\right]$$

$x_3 = \dot{x}_1, \quad x_4 = \dot{x}_2$

$$\overbrace{\frac{d}{dt}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}}^{\dot{\vec{x}}} = \overbrace{\begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\frac{k_2}{M_1} & \frac{k_2}{M_1} & -\frac{c}{M_1} & 0\\ \frac{k_2}{M_2} & \frac{-k_2-k_1}{M_2} & 0 & 0\end{bmatrix}}^{A}\overbrace{\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}}^{\vec{x}} + \overbrace{\begin{bmatrix}0\\0\\0\\1/M_2\end{bmatrix}}^{B}u$$

$$\Rightarrow \dot{\vec{x}} = A\vec{x} + B\vec{u}$$

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