2.5: extension to several Random variable.

Def 3:

$\mathcal{C}$: sample space of the experiment $c \in \mathcal{C}$,

$X_1, \ldots, X_n$: Random variable such that $X_1(c) = x_1$

$X_2(c) = x_2$

$\vdots$

$X_n(c) = x_n$.

$\mathcal{A} = \{ (x_1, \ldots x_n) ; X_1(c) = x_1, \ldots, X_n(c) = x_n \} \subset \mathbb{R}^n$

$A \subset \mathcal{A}$

$P(A) = Pr((X_1, \ldots, X_n) \in A) = P(C)$

$C = \{ c \in \mathcal{C} : X_1(c) = x_1, \ldots, X_n(c) = x_n \}$

$(X_1, \ldots, X_n) \in A$.

Remark:

1. $X_1, \ldots, X_n$ have joint p.d.f $f(x_1, \ldots, x_n)$, $A \subset \mathbb{R}^n$

$$P(A) = \begin{cases} \int_A \cdots \int f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n, & X_1, \ldots, X_n \text{ continuous r.v.} \\ \sum_A \cdots \sum f(x_1, \ldots, x_n), & X_1, \ldots, X_n \text{ discrete r.v.} \end{cases}$$

2. marginal density of X_2 :

$$f_2(x_2) = \begin{cases} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n)\, dx_1\, dx_3 \cdots dx_n , & \text{continuous.} \\ \sum_{x_n=-\infty}^{\infty} \cdots \sum_{x_3=-\infty}^{\infty} \sum_{x_1=-\infty}^{\infty} f(x_1, \ldots, x_n) , & x_1, \ldots, x_n \text{ discrete.} \end{cases}$$

3. marginal joint density of X_3, X_4, X_7 .

$$f_{347}(x_3, x_4, x_7) = \begin{cases} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n)\, dx_1 dx_2 dx_5 dx_6 dx_8 \cdots dx_n , & x_i \text{ continuous} \\ \sum_{x_n} \sum_{x_8} \sum_{x_6} \sum_{x_5} \sum_{x_2} \sum_{x_1} f(x_1, \ldots, x_n) : & , x_i \text{ discrete.} \end{cases}$$

4. conditional density of X_3, X_4, X_7 given $X_2 = x_2$.

$$f_{347|2}(x_3, x_4, x_7 \mid x_2) = \frac{f_{2347}(x_2, x_3, x_4, x_7)}{f_2(x_2)} , \quad f_2(x_2) \neq 0 .$$

5. conditional density of X_3, X_4, X_7 given $X_1 = x_1$, $X_2 = x_2$.

$$f_{347|12}(x_3, x_4, x_7 \mid x_1, x_2) = \frac{f_{12347}(x_1, x_2, x_3, x_4, x_7)}{f_{12}(x_1, x_2)} , \quad f_{12}(x_1, x_2) \neq 0$$

6. CDF of $X_1, \ldots, X_n$

$$F(x_1, \ldots, x_n) = \Pr(X_1 \le x, \ldots X_n \le x_n)$$

$$= \begin{cases} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(t_1, \ldots, t_n)\, dt_1 \cdots dt_n & , \quad X_i \text{ continuous.} \\ \sum_{t_n \le x_n} \cdots \sum_{t_1 \le x_1} f(t_1, \ldots, t_n) & , \quad X_i \text{ discrete.} \end{cases}$$

7. X_i continuous,

$$\frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n} F(x_1, \ldots, x_n) = f(x_1, \ldots, x_n).$$

8. $$E(u(x_1, \ldots, x_n)) = \begin{cases} \int \cdots \int u(x_1, \ldots, x_n)\, dx_1, \ldots, dx_n & , \quad X_i \text{ cont.} \\ \sum_{x_n} \cdots \sum_{x_1} u(x_1, \ldots, x_n)\, f(x_1, \ldots, x_n) & , \quad X_i \text{ discrete.} \end{cases}$$

Def: $X_1, \ldots, X_n$ Random variables

1. $X_1, \ldots, X_n$ Pairwise independent,

$f_{ij}(x_i, x_j) = f_i(x_i) \cdot f_j(x_j)$, $\forall i, j = 1, \ldots, n$, $i \neq j$.

2. $X_1, \ldots, X_n$ (Mutually) independent

$f_{i_1, i_2, \ldots, i_k}(x_{i_1}, \ldots, x_{i_k}) = f_{i_1}(x_{i_1}) \cdots f_{i_k}(x_{i_k})$

where $i_1, \ldots, i_k$ distinct integers from $\{1, \ldots, n\}$.

Remark:

$X_1, \dots, X_n$ independent $\Rightarrow$ $X_1, \dots, X_n$ Pairwise indep.

Remark:

$X_1, \dots, X_n$ independent $\Rightarrow Pr(a_1 < X_1 < b_1, \dots, a_n < X_n < b_n) = Pr(a_1 < X_1 < b_1) \cdots Pr(a_n < X_n < b_n)$.

Remark:

$X_1, \dots, X_n$ independent $\Leftrightarrow M(t_1, \dots, t_n) = M_1(t_1) \cdots M_n(t_n)$, $-h_1 < t_1 < h_1, \dots, -h_n < t_n < h_n$

$h_1 > 0, \dots, h_n > 0$.

Remark:

$X_1, \dots, X_n$ independent $\Rightarrow E(u_1(X_1) \cdot u_2(X_2) \cdots u_n(X_n)) = E(u_1(X_1)) \cdots E(u_n(X_n))$.

exp1:

$$f(x,y,z) = \begin{cases} e^{-x-y-z} & , x>0, y>0, z>0 \\ 0 & , \text{elsewhere} \end{cases} \quad , \quad \text{CDF:}$$

$$F(x,y,z) = Pr(X \le x, Y \le y, Z \le z) = \int_{-\infty}^{z}\int_{-\infty}^{y}\int_{-\infty}^{x} f(t_1,t_2,t_3)\, dt_1\, dt_2\, dt_3$$

$$= \int_0^z\int_0^y\int_0^x e^{-t_1} e^{-t_2} e^{-t_3}\, dt_1\, dt_2\, dt_3$$

$$= (1-e^{-x})(1-e^{-y})(1-e^{-z})$$

$$\Rightarrow F(x,y,z) = \begin{cases} (1-e^{-x})(1-e^{-y})(1-e^{-z}) & , x \ge 0, y \ge 0, z \ge 0 \\ 0 & , \text{elsewhere} \end{cases}$$

exp2 : X_1, X_2, X_3 mutually indep. random Variable with p.d.f

$$f(x) = \begin{cases} 2x & , \; 0 < x < 1 \\ 0 & , \; \text{elsewhere} \end{cases}$$

1. joint p.d.f of X_1, X_2, X_3.

$$f_{123}(x_1, x_2, x_3) = \begin{cases} 8x_1x_2x_3 & , \; 0 < x_1 < 1, \; 0 < x_2 < 1, \; 0 < x_3 < 1 \\ 0 & , \; \text{elsewhere.} \end{cases}$$

2. $E(5X_1X_2^2 + 3X_2X_3^4) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (5x_1x_2^2 + 3x_2x_3^4)\, f_{123}(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3$

$$= \int_0^1\int_0^1\int_0^1 (5x_1x_2^2 + 3x_2x_3^4)(8x_1x_2x_3)\, dx_1\, dx_2\, dx_3$$

$$= 2.$$

3. Define $Y = \max(X_1, X_2, X_3)$. $Pr(Y \le \frac{1}{2})$?

$Pr(Y \le \frac{1}{2}) = Pr(\max(X_1, X_2, X_3) \le \frac{1}{2})$

$= Pr(X_1 \le \frac{1}{2}, X_2 \le \frac{1}{2}, X_3 \le \frac{1}{2})$

idep. $= Pr(X_1 \le \frac{1}{2})\, Pr(X_2 \le \frac{1}{2})\, Pr(X_3 \le \frac{1}{2})$

have same pd.f $= \int_{-\infty}^{\frac{1}{2}} f(x)\, dx \times \int_{-\infty}^{\frac{1}{2}} f(x)\, dx \times \int_{-\infty}^{\frac{1}{2}} f(x)\, dx$

$$= \left(\int_{-\infty}^{\frac{1}{2}} 2x\, dx \right)^3 = \left(\frac{1}{4}\right)^3 = \frac{1}{64}$$

4. Find the p.d.f of Y.

$G(y)$ = c.d.f of Y

$g(y)$ = p.d.f of y

$$G(y) = Pr(Y \le y) = Pr(\max(X_1, X_2, X_3) \le y)$$

$$= Pr(X_1 \le y, X_2 \le y, X_3 \le y)$$

$$\underset{\text{indep}}{=} Pr(X_1 \le y)\, Pr(X_2 \le y)\, Pr(X_3 \le y)$$

$$= \int_{-\infty}^{y} f(x)\,dx \int_{-\infty}^{y} f(x)\,dx \int_{-\infty}^{y} f(x)\,dx$$

$$= \left(\int_{-\infty}^{y} f(x)\,dx\right)^3$$

$$G(y) = \begin{cases} 0 & , y \le 0 \\ (y^2)^3 = y^6 & , 0 \le y \le 1 \\ (1)^3 = 1 & , 1 \le y \end{cases}$$

$\Rightarrow g(y) = \bar{G}(y)$ for all y where $g(y)$ is cont.

$$g(y) = \begin{cases} 6y^5 & , 0 < y < 1 \\ 0 & , \text{elsewhere.} \end{cases}$$

Done of chapter 2.