7.5 Indeterminate forms and L'Hopital's rule.

10 + 16 + 21 + 27 + 30 34 + 38 + 41 + 46 + 52 + 56 + 60 + 62 + 66 + 68 + 72

10.

$$\lim_{t \to 1} \frac{t^3 - 1}{4t^3 - t - 3} \left(\frac{0}{0}\right) = \lim_{t \to 1} \frac{3t^2}{12t^2 - 1} = \boxed{\frac{3}{11}}$$

16.

$$\lim_{x \to 0} \frac{\sin x - x}{x^3} \quad \frac{0}{0}$$

$$\lim_{x \to 0} \frac{\cos x - 1}{3x^2} \quad \frac{0}{0}$$

$$\lim_{x \to 0} \frac{-\sin x}{6x} \quad \left(\frac{0}{0}\right)$$

$$\lim_{x \to 0} \frac{-\cos x}{6} = \boxed{-\frac{1}{6}}$$

21.

$$\lim_{x \to 0} \frac{x^2}{\ln(\sec x)} \quad \left(\frac{0}{0}\right)$$

$$\lim_{x \to 0} \frac{2x}{\frac{\sec x \tan x}{\sec x}} = \lim_{x \to 0} \frac{2x}{\tan x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2}{\sec^2 x} = \frac{2}{1} = \boxed{2}$$

27 $\lim_{\theta \to 0} \frac{3^{\sin\theta} - 1}{\theta} \quad \left(\frac{0}{0}\right)$

$\lim_{\theta \to 0} \frac{3^{\sin\theta} \cos\theta \ln 3}{1} = 3^{0} \cdot (\cos 0) \ln 3$

$= \boxed{\ln 3}$

30 $\lim_{x \to 0} \frac{3^x - 1}{2^x - 1} \quad \left(\frac{0}{0}\right)$

$= \lim_{x \to 0} \frac{3^x \cdot \ln 3}{2^x \cdot \ln 2} = \boxed{\frac{\ln 3}{\ln 2}}$ (Note that $\frac{\ln 3}{\ln 2} \neq \ln 3 - 2$)

34 $\lim_{x \to 0^+} \frac{\ln(e^x - 1)}{\ln x} \qquad \frac{\ln 0^+}{\ln 0^+}$

$= \lim_{x \to 0^+} \frac{\frac{e^x}{e^x - 1}}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{x e^x}{e^x - 1} \quad \left(\frac{0}{0}\right)$

$= \lim_{x \to 0^+} \frac{x e^x + e^x \cdot 1}{e^x} = \frac{1}{1} = \boxed{1}$

38 $\lim_{x\to 0^+} (\ln x - \ln \sin x)$ $(-\infty - -\infty = \infty - \infty)$

3

$\lim_{x\to 0^+} \ln\left(\frac{x}{\sin x}\right)$

$= \ln\left[\lim_{x\to 0^+} \frac{x}{\sin x}\right]$

$= \ln\left(\lim_{x\to 0^+} \frac{1}{\cos x}\right) = \ln 1 = \boxed{0}$

41 $\cdot \lim_{x\to 1^+} \frac{1}{x-1} - \frac{1}{\ln x}$ $\infty - \infty$

$\lim_{x\to 1^+} \frac{\ln x - (x-1)}{(x-1)\ln x}$

$\lim_{x\to 1^+} \frac{\ln x - x + 1}{(x-1)\ln x}$ $\frac{0}{0}$

$\lim_{x\to 1^+} \frac{\frac{1}{x} - 1}{(x-1)\cdot\frac{1}{x} + \ln x} = \lim_{x\to 1^+} \frac{\frac{1}{x} - 1}{1 - \frac{1}{x} + \ln x}$ $\left(\frac{0}{0}\right)$

$= \lim_{x\to 1^+} \frac{\frac{-1}{x^2}}{\frac{1}{x^2} + \frac{1}{x}} = \frac{-1}{1+1} = \boxed{\frac{-1}{2}}$

(46) $\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty} \frac{x^2}{e^x} \quad \left(\frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \frac{2x}{e^x} \quad \left(\frac{\infty}{\infty}\right)$

$= \lim_{x\to\infty} \frac{2}{e^x} = \boxed{0}$

(52) $\cdot \lim_{x\to 1^+} x^{\frac{1}{x-1}} \qquad 1^\infty$

$y = x^{\frac{1}{x-1}}$

$\ln y = \frac{1}{x-1} \ln x = \frac{\ln x}{x-1}$

$\lim_{x\to 1^+} \ln y = \lim_{x\to 1^+} \frac{\ln x}{x-1} \quad \left(\frac{0}{0}\right)$

$= \lim_{x\to 1^+} \frac{\frac{1}{x}}{1} = 1$

So $\lim_{x\to 1^+} x^{\frac{1}{x-1}} = \boxed{e^1}$

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56 $\lim_{x\to\infty} \cdot x^{\frac{1}{\ln x}}$ ∞^0

$y = x^{\frac{1}{\ln x}}$

$\ln y = \ln x^{\frac{1}{\ln x}}$

$\ln y = \frac{1}{\ln x} \cdot \ln x = 1$

$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} 1 = 1$

So $\lim_{x\to\infty} x^{\frac{1}{\ln x}} = \boxed{e^1}$

60 $\lim_{x\to 0^+} \left(1+\frac{1}{x}\right)^x = \boxed{e^1}$

62 $\lim_{x\to\infty} \left(\frac{x^2+1}{x+2}\right)^{\frac{1}{x}}$ ∞^0

$y = \left(\frac{x^2+1}{x+2}\right)^{\frac{1}{x}} \longrightarrow \ln y = \frac{\ln\left(\frac{x^2+1}{x+2}\right)}{x}$ $\left(\frac{\infty}{\infty}\right)$

$$= \lim_{x\to\infty} \frac{\dfrac{(x+2)[2x]-(x^2+1)\cdot 1}{(x+2)^2 \left(\dfrac{x^2+1}{x+2}\right)}}{1} = \lim_{x\to\infty} \frac{2x^2+4x-x^2-1}{\dfrac{x^2+1}{x+2}\,(x+2)^2}$$

$$\cancel{= \lim_{x\to\infty} \frac{(x^2-4x-1)}{(x^2+1)}\, x+2}$$

$$= \lim_{x\to\infty} \frac{2x^2+4x-x^2-1}{(x+2)(x^2+1)} = \lim_{x\to\infty} \frac{x^2+4x-1}{x^3+x+2x^2+2}$$

$$= \lim_{x\to\infty} \frac{x^2+4x-1}{x^3+2x^2+x+2} = 0$$

$$\text{So } \lim_{x\to 0^+} \left(1+\frac{1}{x}\right)^x = e^0 = \boxed{1}$$

66 $\lim_{x \to 0^+} \sin x \ln x$ $\quad 0 \cdot -\infty$

$$\lim_{x \to 0^+} \frac{\ln x}{\frac{1}{\sin x}} = \lim_{x \to 0^+} \frac{\ln x}{\csc x}$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\csc x \cot x}$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}} = \lim_{x \to 0^+} \cancel{\frac{-\sin^2 x \cos x}{x}}$$

$$= \lim_{x \to 0^+} \frac{-\sin x \tan x}{x} = \boxed{0}$$

68 $$\lim_{x \to 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}} = \lim_{x \to 0^+} \sqrt{\frac{x}{\sin x}} = \sqrt{\lim_{x \to 0^+} \frac{x}{\sin x}} = 1$$

72 $$\lim_{x \to -\infty} \frac{2^x + 4^x}{5^x - 2^x} = \lim_{x \to -\infty} \frac{1 + 2^x}{\left(\frac{5}{2}\right)^x - 1} = \boxed{-1}$$