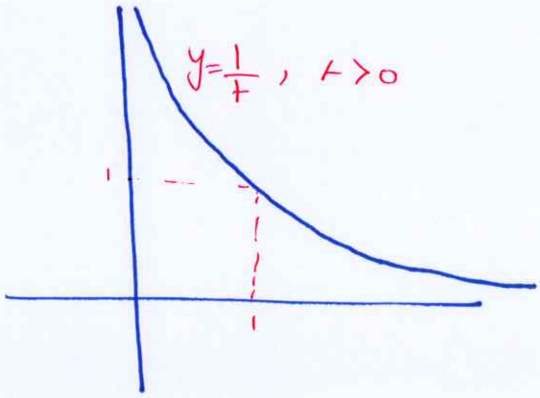
7.2. Natural Logarithm Function

Def. The Natural logarithm function defined as

$$\boxed{\ln x = \int_1^x \frac{1}{t}\,dt \quad , \quad x > 0}$$



Some properties of the function $f(x) = \ln x$

$\boxed{1}$ If $x > 1 \longrightarrow \ln x = \int_1^x \frac{1}{t}\,dt$ = The positive area of the graph $y=\frac{1}{t}$ from $t=1 \to t=x$

$\boxed{2}$ If $0 < x < 1 \longrightarrow \ln x = \int_1^x \frac{1}{t}\,dt = -\int_x^1 \frac{1}{t}\,dt$ = the negative Area of the graph $y=\frac{1}{t}$ from $t=x$ to $t=1$

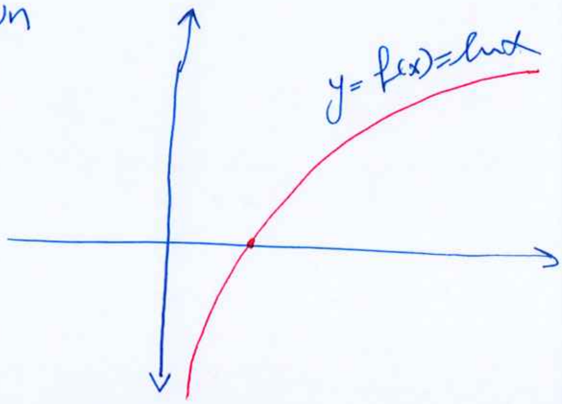
$\boxed{3}$ If $x = 1 \longrightarrow \ln x = \int_1^1 \frac{1}{t}\,dt = 0$

$\boxed{4}$ $\frac{d}{dx}(\ln x) = \frac{d}{dx}\left(\int_1^x \frac{1}{t}\,dt\right) = \frac{1}{x} > 0$ for all $x > 0$

Then $f(x) = \ln x$ is increasing fn. on $\mathbb{R}$

$\boxed{5}$ $\frac{d^2}{dx^2}(\ln x) = \frac{d}{dx}\left(\frac{1}{x}\right) = \frac{-1}{x^2}$, $\forall x > 0$

Then $f(x) = \ln x$ is concave down



$\boxed{6}$ $\text{Domain}_{\ln x} = (0, \infty)$

$\text{Range}_{\ln x} = (-\infty, \infty)$

$\boxed{7}$ From the graph of $y = \ln x$. We have.

$\lim_{x \to \infty} \ln x = \infty$ $\qquad$ $\lim_{x \to 0^+} \ln x = -\infty$ $\qquad$ $\lim_{x \to 0} \ln x = \text{D.N.E.}$

because the function is undefined at $x=0$

$\boxed{8}$ If $x > 0$, and $y > 0$, then

$\ln xy = \ln x + \ln y$

$\ln \frac{x}{y} = \ln x - \ln y$

$\ln \left(\frac{1}{x}\right) = -\ln x$

$\ln x^n = n \ln x$

9 If U is a differentiable fn. of x.

$$\frac{d}{dx} \ln(u) = \frac{1}{U} \frac{dU}{dx} = \frac{U'}{U}$$

9 $\int \frac{1}{x} \, dx = \ln|x| + C$

Since $\frac{d}{dx} \ln x = \frac{1}{x}$, $x > 0$

$\frac{d}{dx} \ln(-x) = \frac{1}{x}$, $x < 0$

In general :- $\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$.

Def The number "e" : is a number in the domain $\ln x$

Such that $\boxed{\ln e = 1}$

$e = 2.718281828182$ - Euler's Number

Exp.1 Simplify the following expression

1 $\frac{\ln 49}{\ln 21 + \ln \frac{1}{3}} = \frac{\ln 7^2}{\ln 7 + \ln 3 + \ln 1 - \ln 3} = \frac{2 \ln 7}{\ln 7} = 2$

2 $3 \ln \sqrt[3]{2} + \frac{1}{2} \ln\left(\frac{9}{4}\right) = \ln 2 + \ln \sqrt{\frac{9}{4}}$

$= \ln 2 + \ln 3 - \ln 2 = \ln 3$

1 Solve for x

$$\ln(x+2)(2x+1) = 2\ln(x+2)$$

Sol: $\ln(x+2) + \ln(2x+1) = \ln(x+2)^2$

$$\ln(2x+1) = \ln(x+2)^2 - \ln(x+2)$$

$$\ln(2x+1) = \ln\frac{(x+2)^2}{(x+2)}$$

$\ln(2x+1) = \ln(x+2)$ Since $y=\ln x$ is increasing
$y=\ln x$ 1-1 function

Then $\longrightarrow$ $2x+1 = x+2$

$$\boxed{x = 1}$$

2 Solve for y

$$\ln(2^y - 1) + \ln 2 = x + \ln 2x$$

$$\ln 2(2^y - 1) = x + \ln 2x$$

$$2(2^y - 1) = e^{x+\ln 2x}$$

$$2(2^y - 1) = 2x\,e^x$$

$(2^y - 1) = x\,e^x \longrightarrow 2^y = xe^x + 1$

$$y = \frac{\ln(xe^x+1)}{\ln 2} = \log_2(xe^x+1)$$

3 Solve for x

$$\ln e + 4^{-2\log_4 x} = \frac{1}{x}\log_{10} 100$$

$$1 + x^{-2} = \frac{2}{x}$$

$$\frac{x^2+1}{x^2} = \frac{2}{x} \longrightarrow x^2 - 2x + 1 = 0$$

$$(x-1)^2 = 0$$

$$\boxed{x = 1}$$

4 Find the Domain of $f(x) = \ln\left(\log_3\left(e^{2x} - 3\right)\right)$

Domain $f(x)$: $\{x : \log_3(e^{2x} - 3) > 0\}$

$\{x : e^{2x} - 3 > 1\}$

$\{x : e^{2x} > 4\}$

$\{x : 2x > \ln 4\}$

$\{x : 2 > \ln 2\}$

5 Find $\frac{dy}{dx}$ for each of the following

1. $y = x\sqrt{\ln x}$

$$\frac{dy}{dx} = \frac{x \cdot \frac{1}{x}}{2\sqrt{\ln x}} + \sqrt{\ln x}$$

$$= \frac{1}{2\sqrt{\ln x}} + \sqrt{\ln x}$$

2. $y = \left(x^2 \ln x\right)^4$

$$\frac{dy}{dx} = 4\left(x^2 \ln x\right)^3 \cdot \left[x^2 \cdot \frac{1}{\cancel{x}} + \ln x \cdot 2x\right]$$

$$= 4\left(x^2 \ln x\right)^3 \left(x + 2x \ln x\right)$$

3. $y = \ln\left(\ln\left(3x+4\right)\right)$

$$\frac{dy}{dx} = \frac{1}{\ln(3x+4)} \cdot \frac{1}{3x+4} \cdot 3 \,.$$

4. $y = \ln\left(\frac{\sqrt{x+1}}{3+x^2}\right)$

$$y = \frac{1}{2}\ln(x+1) - \ln\left(3+x^2\right)$$

$$\frac{dy}{dx} = \frac{1}{2(x+1)} - \frac{1}{3+x^2} \cdot 2x$$

5 $y = \ln\left[\sin(3x)\sqrt{1+x^2}\right]$

$$y = \ln \sin(3x) + \frac{1}{2}\ln(1+x^2)$$

$$\frac{dy}{dx} = \frac{3\cos(3x)}{\sin(3x)} + \frac{1}{2}\,\frac{2x}{1+x^2}$$

$$\frac{dy}{dx} = 3\cot(3x) + \frac{x}{1+x^2}$$

6 $y = \int_{x^2}^{x^3} t \ln t \; dt$

$$\frac{dy}{dx} = x^3 \ln x^3 (3x^2) - x^2 \ln x^2 (2x)$$

$$= 3x^5 \ln x^3 - 2x^3 \ln x^2$$

$$= 9x^5 \ln x - 4x^3 \ln x$$

7 $y = x^{\tan x}$

$$\ln y = \tan x \ln x$$

$$\frac{y'}{y} = \tan x \cdot \frac{1}{x} + \ln x \cdot \sec^2 x$$

$$y' = \left[\frac{\tan x}{x} + \sec^2 x \ln x\right] x^{\tan x}$$

Q64 page 376

$$y = \frac{x \sin x}{\sqrt{\sec x}}$$

$$\ln y = \ln x + \ln \sin x - \frac{1}{2} \ln \sec x$$

$$\frac{y'}{y} = \frac{1}{x} + \frac{\cos x}{\sin x} - \frac{1}{2} \frac{\sec x \tan x}{\sec x}$$

$$y' = \left[\frac{1}{x} + \cot x - \frac{1}{2} \tan x\right] \frac{x \sin x}{\sqrt{\sec x}}$$

Evaluate the following integral :-

$$* \int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\ln|\cos x| + C$$

$$= \ln|\sec x| + C$$

$$* \int \cot x \, dx = \int \frac{\cos x}{\sin x} dx = \ln|\sin x| + C$$

$$* \int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx = \ln|\sec x + \tan x| + C$$

$$* \int \csc x \, dx = \int \frac{\csc x(\csc x + \cot x)}{\csc x + \cot x} = -\ln|\csc x + \cot x| + C$$

$$= \ln|\csc x - \cot x| + C$$

(2) $\int \frac{x+3}{x^2+6x+5}\,dx$

$U = x^2+6x+5$

$dU = 2x+6\,dx$

$= \frac{1}{2}\int \frac{dU}{U}$

$= \frac{1}{2}\ln|U| + C$

$= \frac{1}{2}\ln|x^2+6x+5| + C$

(3) $\int \frac{1}{x\cos^2(8+\ln x)}\,dx$

$U = 8+\ln x$

$dU = \frac{1}{x}\,dx$

$= \int \frac{dU}{\cos^2(U)}$

$= \int \sec^2(U)\,dU = \tan(U) + C$

$= \tan(8+\ln x) + C$

(4) $\int_2^{16} \frac{dx}{2x\sqrt{\ln x}}$

$U = \ln x$

$dU = \frac{1}{x}\,dx$

$x = 2 \longrightarrow U = \ln 2$

$x = 16 \longrightarrow U = \ln 16$

$\frac{1}{2}\int_{\ln 2}^{\ln 16} \frac{dU}{\sqrt{U}} = \frac{1}{2}.2\,U^{1/2}\Big]_{\ln 2}^{\ln 16}$

$= \frac{2}{2}\left[\sqrt{\ln 16} - \sqrt{\ln 2}\right] = \frac{2}{2}\left[2\sqrt{\ln 2} - \sqrt{\ln 2}\right]$

$= \sqrt{\ln 2}$

$\boxed{6}$ $$\int \frac{dx}{\sqrt{x}+x} = \int \frac{1}{\sqrt{x}(1+\sqrt{x})}\,dx$$

$U = 1+\sqrt{x}$

$dU = \frac{1}{2\sqrt{x}}\,dx$

$$= \int \frac{2\,dU}{U}$$

$$= 2\ln|U| + C$$

$$= 2\ln|1+\sqrt{x}| + C$$

$\boxed{7}$ $$\int \frac{\sec x}{\sqrt{\ln(\sec x+\tan x)}}\,dx$$

$U = \ln(\sec x + \tan x)$

$dU = \sec x\,dx$

$$= \int \frac{\cancel{\sec x}}{\sqrt{U}}\,\frac{dU}{\cancel{\sec x}}$$

$$= 2U^{1/2} + C$$

$$= 2\sqrt{\ln(\sec x+\tan x)} + C$$

8 $$\int \frac{\ln x}{(2+\ln x)\,x}\,dx$$

$U = 2+\ln x$

$dU = \frac{1}{x}\,dx$

$$\int \frac{U-2}{U}\,dU = U - 2\ln|U| + C$$

$$= 2+\ln x - 2\ln|2+\ln x| + C = \ln x - 2\ln|2+\ln x| + C$$

$\boxed{9}$ $\int \cot x \ln(\sin x)\, dx$

$u = \ln \sin x$

$du = \cot x \, dx$

$= \int u \, du$

$= \dfrac{u^2}{2} + C$

$= \dfrac{1}{2} \ln^2 \sin x + C$

Q_{70} page $_{376}$ Prove that $f(x) = x - \ln x$ is increasing for $x > 1$

Show that $\ln x < x$ if $x > 1$

$f'(x) = 1 - \dfrac{1}{x} = \dfrac{x-1}{x} > 0$ for $x > 1$

$\longrightarrow f(x)$ increasing on $(1, \infty)$

$f(x)$ is increasing for $x > 1$ [If $x_1 > x_2 \rightarrow f(x_1) > f(x_2)$]

$x > 1$

Then $\rightarrow$ $f(x) > f(1)$

$x - \ln x > 1 - \ln 1$

$x - \ln x > 1 > 0$

$\boxed{x > \ln x}$ for $x > 1$