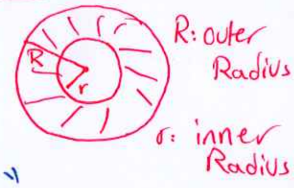
Chapter 6:- Application of Definite Integrals:-

6.1 : Volume Using Cross-Sections:-

- Disk Method "with Rotation"

- Washer Method "with Rotation"

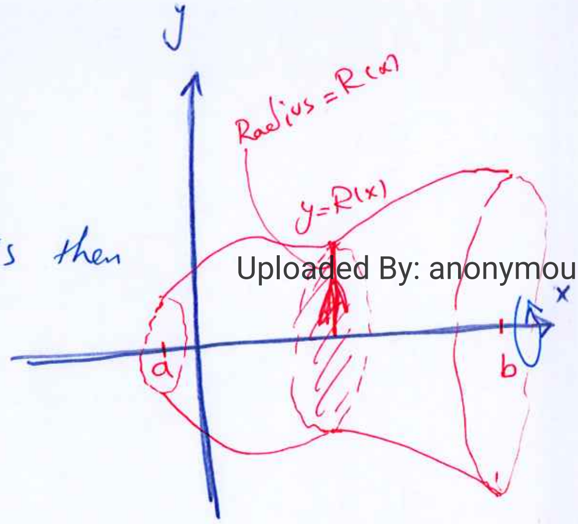


- Slicing Method "without Rotation"

[1] Volume of Solid of revolution "The Disk Method"

The Volume of the solid generated by revolving the region between the graph of the fn. $y = R(x)$ and the x-axis about x-axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi\,[R(x)]^2\,dx$$



If the revolving about the y-axis then

$$V = \int_c^d A(y)\,dy = \int_c^d \pi\,[R(y)]^2\,dy$$

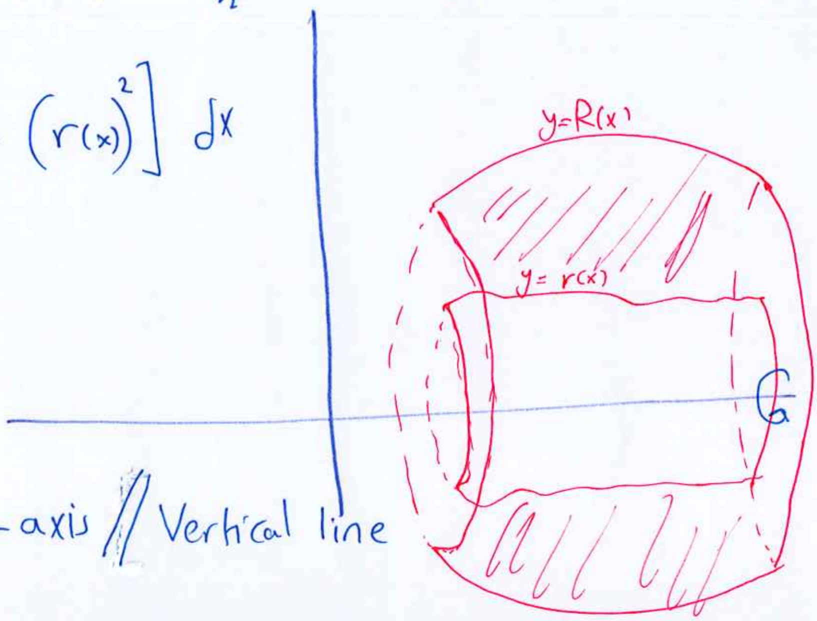
[2] Volume of Solid of revolution "The Washer Method"

We use this Method if the solid of revolution has a hole in it.

$$V = \int_a^b \pi\left[(R(x))^2 - (r(x))^2\right]dx$$ where $R(x)$: Outer Radius, $r(x)$: Inner Radius

* If the revolving about X-axis // horizontal line

$$V=\int_a^b A(x)\,dx=\int_a^b \pi\left[(R(x))^2-(r(x))^2\right]dx$$



* If the revolving about y-axis // Vertical line

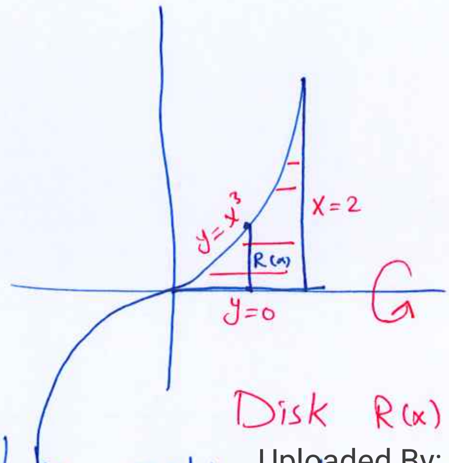
$$V=\int_c^d A(y)\,dy=\int_c^d \pi\left[(R(y))^2-(r(y))^2\right]dy$$

[1] Find the volume of the solid generated by revolving the region bounded by $y=x^3$, $y=0$, $x=2$ about the x-axis

$$V=\int_0^2 A(x)\,dx=\int_0^2 \pi\left[R(x)\right]^2dx$$

$$=\pi\int_0^2 (x^3)^2\,dx=\pi\frac{x^7}{7}\Big]_0^2$$

$$=\frac{128\pi}{7}$$



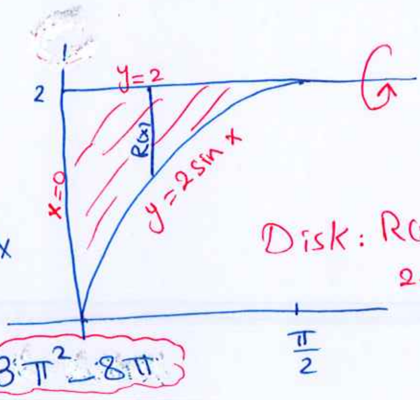
Disk $R(x)=x^3$

[2] find the volume of the solid generated by revolving the region in the first quadrant bounded above by $y=2$, below by $y=2\sin x$ $0\leqslant x\leqslant\frac{\pi}{2}$ and on the left by the y-axis about $y=2$

$$V=\int_0^{\pi/2} A(x)\,dx=\int_0^{\pi/2}\pi\left[2-2\sin x\right]^2dx$$

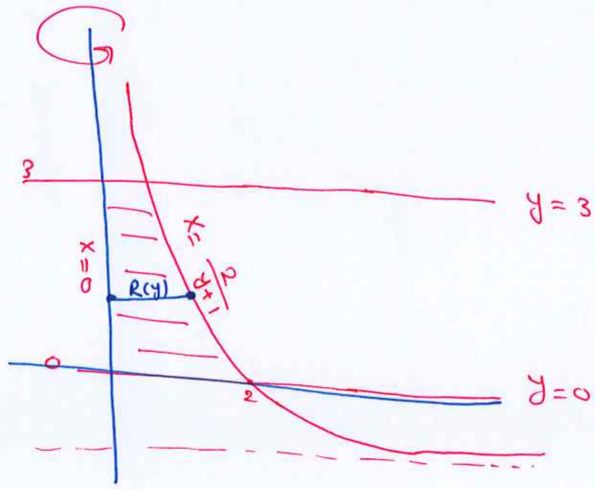
$$=\pi\int_0^{\pi/2} 4-8\sin x+4\sin^2 x\;dx$$

$$=\pi\left[4x+8\cos x+2x-\sin(2x)\right]_0^{\pi/2}=3\pi^2-8\pi$$



Disk: $R(x)=\cancel{2-s}$ $2-2\sin x$

$\boxed{3}$ Find the volume of the solid generated by revolving the region bounded by the $y=0$, $x=0$, $x=\frac{2}{y+1}$, $y=3$ about the y-axis



Disk. $R(y) = \frac{2}{y+1}$

$$V = \int_0^3 A(y)\,dy$$

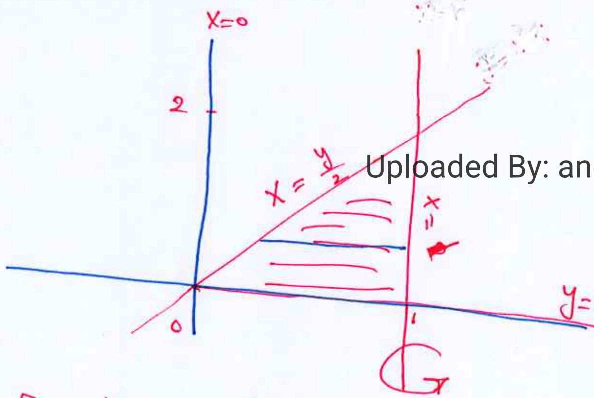
$$= \int_0^3 \pi \left[\frac{2}{y+1}\right]^2 dy$$

$$= \int_0^3 \frac{4\pi}{(y+1)^2}\,dy$$

$$= 4\pi \left.\frac{(y+1)^{-1}}{-1}\right]_0^3$$

$$= -4\pi\left[\frac{1}{4} - 1\right] = 3\pi$$

$\boxed{4}$ Find the volume of the solid generated by revolving the region bounded by the line $y=2x$, $y=0$, and $x=1$ about $x=1$



Disk. $R(y) = 1 - \frac{y}{2}$

$$V = \int_0^2 A(y)\,dy$$

$$= \int_0^2 \pi\left[R(y)\right]^2 dy$$

$$= \int_0^2 \pi\left[1 - \frac{y}{2}\right]^2 dy$$

$$\int_0^2 \pi\left[\frac{2-y}{2}\right]^2 dy = \frac{\pi}{4}\left.\frac{(2-y)^3}{3(-1)}\right]_0^2$$

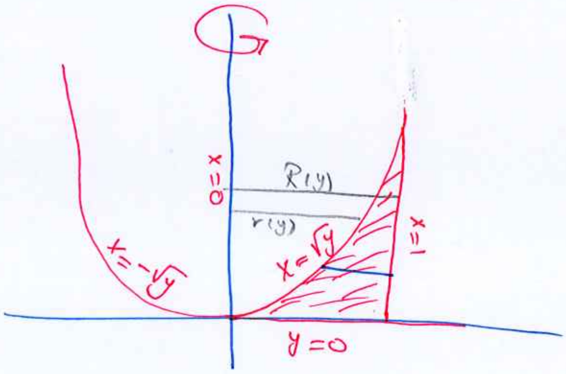
$$= \frac{\pi}{-12}\left[0 - 8\right] = \frac{2\pi}{3}$$

$\boxed{5}$ Find the volume of the solid generated by revolving the region bounded by the Parabola $y=x^2$, below by x_axis and on the right by the line $x=1$

I: $\boxed{\text{about } y\text{-axis}}$

II: $\boxed{\text{about line } x=2}$

I. about y-axis



$$V=\int_0^1 A(y)\,dy$$

$$=\int_0^1 \pi\left[R^2(y)-r^2(y)\right]dy$$

Washer Method

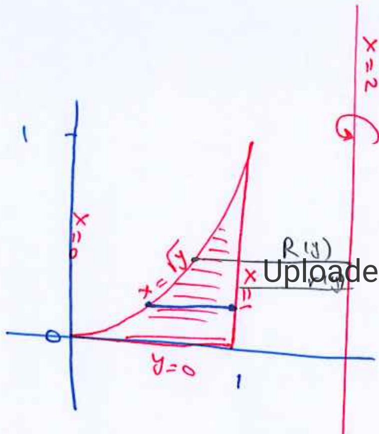
$R(y)=1$

$r(y)=\sqrt{y}$

$$=\int_0^1 \pi\left[1-y\right]dy$$

$$=\pi\left[y-\frac{y^2}{2}\right]_0^1=\frac{\pi}{2}$$

II. about $x=2$



$$V=\int_0^1 A(y)\,dy$$

$$=\int_0^1 \pi\left[4-4\sqrt{y}+y-1\right]dy$$

Washer Method :-

$R(y)=2-\sqrt{y}$

$r(y)=1$

$$=\pi\left[4y-\frac{8}{3}y^{3/2}+\frac{y^2}{2}-y\right]_0^1$$

$$=\pi\left[3y-\frac{8}{3}y^{3/2}+\frac{y^2}{2}\right]_0^1=\frac{5\pi}{6}.$$

6 Find the volume of the solid generated by revolving the region bounded by the curve $y = 4 - x^2$ and the line $y = 2 - x$ about x_axis (Q_{38})

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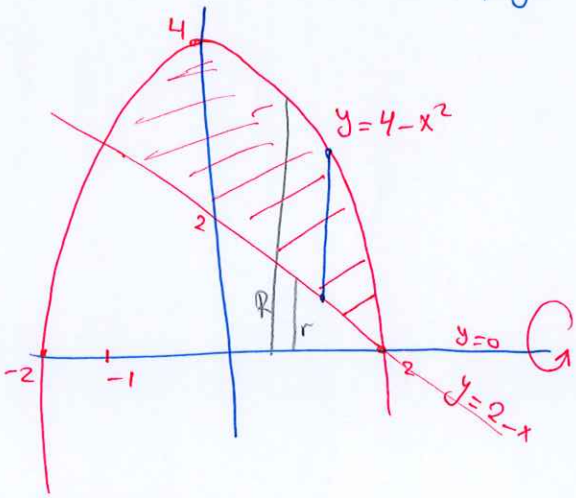
$$V = \int_{-1}^{2} A(x)\, dx$$

$$= \int_{-1}^{2} \pi \left[R^2(x) - r^2(x)\right] dx$$

$$\int_{-1}^{2} \pi \left[16 - 8x^2 + x^4 - [4 - 4x + x^2]\right] dx$$

$$\pi \left[\frac{x^5}{5} - 3x^3 + 2x^2 + 12x\right]_{-1}^{2}$$

$$= \frac{108\pi}{5}$$



Washer

$R(x) = 4 - x^2$

$r(x) = 2 - x$

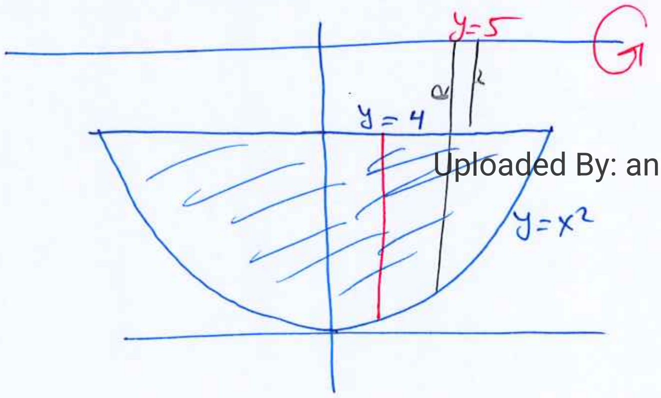
7 Find the volume of the solid generated by revolving the region bounded by $y = x^2$, $y = 4$ about $y = 5$

$$V = \int_{-2}^{2} A(x)\, dx$$

$$= 2\int_{0}^{2} \pi \left([5 - x^2]^2 - 1\right) dx$$

$$= 2\pi \int_{0}^{2} 25 - 10x^2 + x^4 - 1 \; dx$$

$$= 2\pi \left[24x - \frac{10x^3}{3} + \frac{x^5}{5}\right]_0^2 = \frac{832\pi}{15}$$



Washer

$R(x) = 5 - x^2$

$r(x) = 1$

Q24 (lecture Problem) page 317

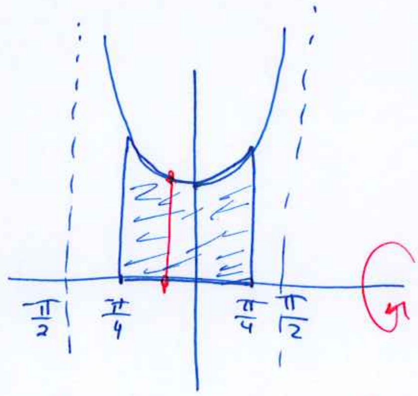
Find Volume if the revolution about x-axis

$y = \sec x \qquad y = 0 \qquad x = \frac{-\pi}{4} \qquad x = \frac{\pi}{4}$

$$V = \int_{-\frac{\pi}{4}}^{\pi/4} A(x)\, dx$$

$$= 2\pi \int_0^{\pi/4} \sec^2 x \quad dx$$

$$= 2\pi \tan x \Big]_0^{\pi/4} = 2\pi$$



Disk $\quad R(x) = \sec x$

Q32 $\quad x = \frac{\sqrt{2y}}{y^2+1}$, $\quad x = 0, \quad y = 1 \qquad$ about y-axis

$$V = \int_0^1 A(y)\, dy = \pi \int_0^1 \frac{2y}{(y^2+1)^2}\, dy$$

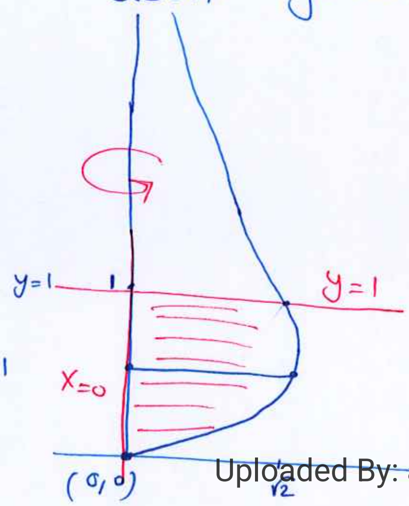
$$= \pi \int_1^2 \frac{du}{u^2} = \frac{\pi}{2}$$

Assume $u = y^2+1$

$du = 2y\, dy$

$y = 0 \longrightarrow u = 1$

$y = 1 \longrightarrow u = 2$

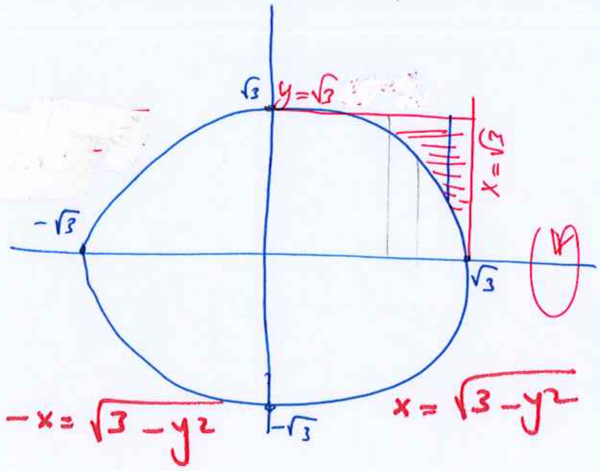


Disk

$R(y) = \frac{\sqrt{2y}}{y^2+1}$

44 About y-axis

$$V = \int_0^{\sqrt{3}} A(y)\, dy$$



$$= \pi \int_0^{\sqrt{3}} 3 - 3 + y^2 \, dy$$

$$= \pi \int_0^{\sqrt{3}} y^2 \, dy = \pi \left. \frac{y^3}{3} \right]_0^{\sqrt{3}}$$

$$= \sqrt{3}\,\pi$$

Washer $R(y) = \sqrt{3}$

$r(y) = \sqrt{3 - y^2}$

3 Volume of Solid without Revolution "Without Rotation"

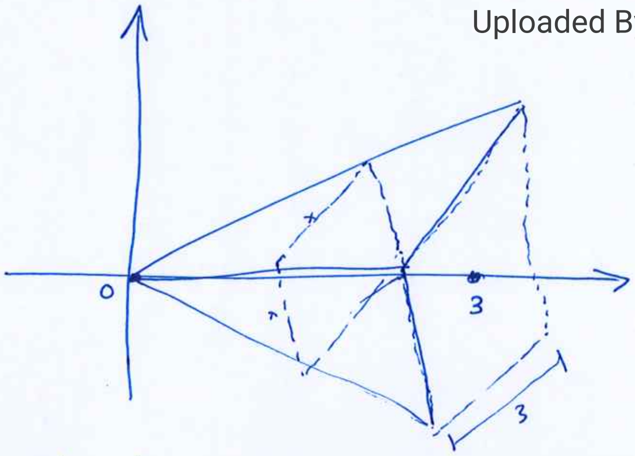
The volume of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ [Cross section perpendicular to x-axis]

$$V = \int_a^b A(x)\, dx$$

Note: The Cross-Section Not Homogeneous

- If the cross section perpendicular to y-axis

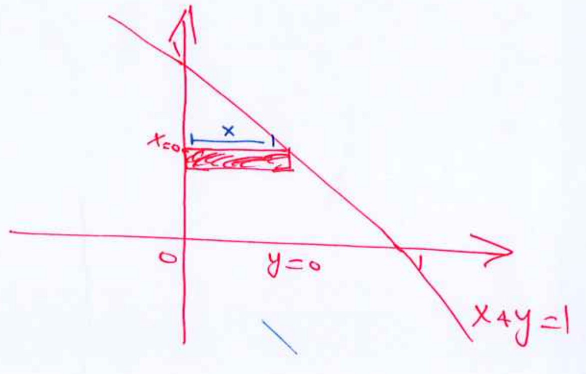
$$V = \int_c^d A(y)\, dy$$



The base of Pyramid is square

$A(x) = x^2$

The base S is triangle enclosed by $x+y=1$ x-axis, y-axis. Find the volume of solid ~~is~~ if cross section perpendicular to y-axis are semicircle.



Sol without Revolution (slicing Method)

$$V=\int_c^d A(y)\,dy$$

$$=\int_0^1 \frac{\pi}{2}\left(\frac{1-y}{2}\right)^2$$

$$\frac{\pi}{8}\ \frac{(1-y)^3}{3(-1)}\Big]_0^1$$

$$\frac{-\pi}{24}\left[0-1\right]=\frac{\pi}{24}.$$

$A(y)$ = Area of semicircle

$$=\frac{1}{2}\pi r^2$$

$$=\frac{1}{2}\pi\left(\frac{x}{2}\right)^2$$

$$=\frac{1}{2}\pi\left(\frac{1-y}{2}\right)^2$$

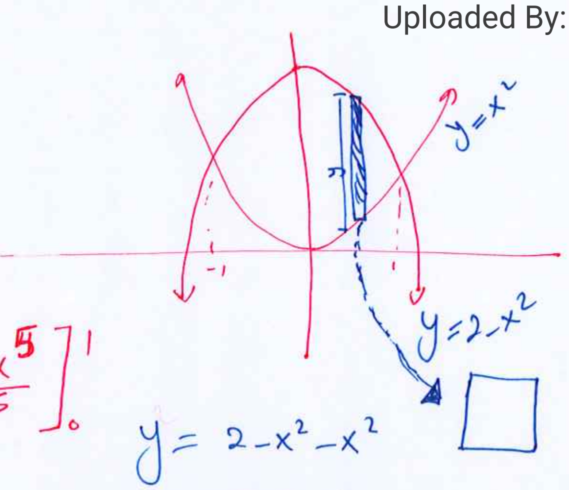
The volume of the solid whose base is bounded by the curve $y=x^2$ and $y=2-x^2$ whose cross section through the solid perpendicular to x-axis are <u>Square</u>

$$V=\int_{-1}^{1} A(x)\,dx \quad \left[\text{Slicing}\right]$$

$$=\int_{-1}^{1}\left(2-2x^2\right)^2\,dx$$

$$=2\int_0^1\left(2-2x^2\right)^2dx=2\left[4x-\frac{8x^3}{3}+\frac{4x^5}{5}\right]_0^1$$

$$=\frac{64}{15}$$

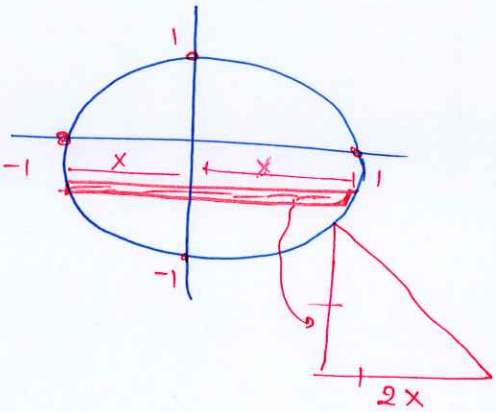


$$y=2-x^2-x^2$$

$$y=2-2x^2$$

The base of the solid is the Disk $x^2+y^2 \leq 1$, the cross section by plane perpendicular to y-axis between $y=-1$, $y=1$ are isosceles right triangle with one leg in the dist

$$V = \int_{\boxed{-1}}^{\boxed{1}} A(y)\, dy$$



$$= 2\int_0^1 2(1-y^2)\, dy$$

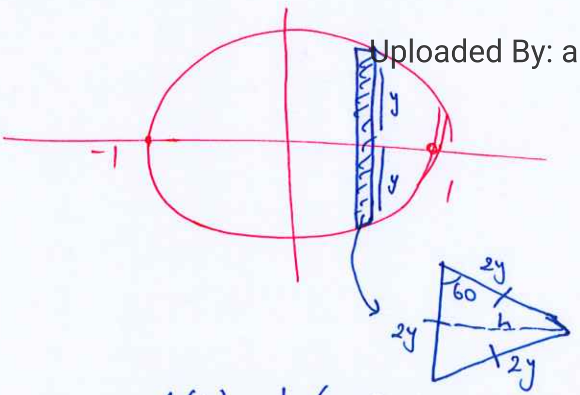
$$A(y) = \frac{1}{2}(2x)(2x)$$

$$= 4\left[y - \frac{y^3}{3}\right]_0^1 = \frac{8}{3}$$

$$= 2x^2$$
$$= 2(1-y^2)$$

The Base S is a circle $x^2+y^2=1$ and the cross section perpendicular to x-axis are equilateral triangle

$$V = \int_{\boxed{-1}}^{\boxed{1}} A(x)\, dx$$



$$\int_{-1}^{1} \sqrt{3}\,(1-x^2)\, dx$$

$$A(x) = \frac{1}{2}(2y)(h)$$

$$= \frac{4}{\sqrt{3}}$$

$$= y\left(\frac{\sqrt{3}}{2} \cdot 2y\right)$$
$$= \sqrt{3}\, y^2 = \sqrt{3}(1-x^2)$$

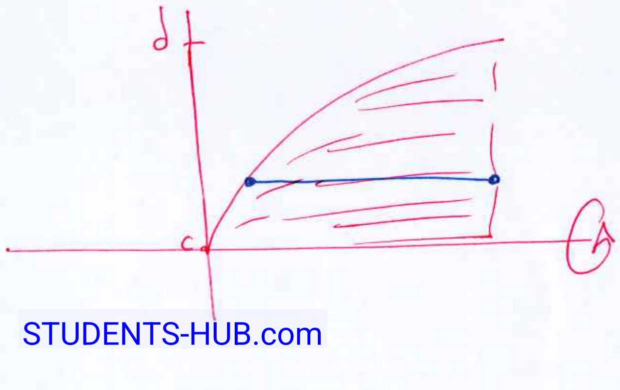
6.2 Volume by Using Cylindrical Shell Method

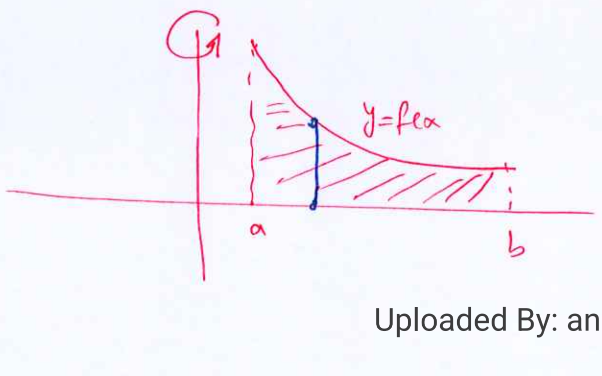
The volume of the solid generated by revolving the region between the x-axis, and the graph of the positive continuous fn $y=f(x)$ about y-axis (or Vertical line)

$$\boxed{V=\int_a^b 2\pi \begin{pmatrix}\text{Shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{Shell}\\ \text{height}\end{pmatrix} dx}$$

If the revolution about x-axis (or horizantal line)

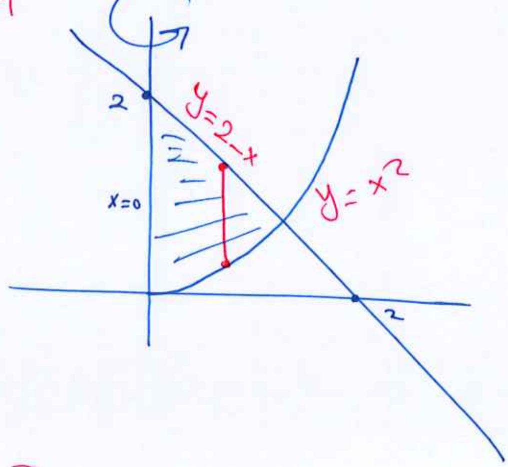
$$\boxed{V=\int_c^d 2\pi \begin{pmatrix}\text{Shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{Shell}\\ \text{Length}\end{pmatrix} dy}$$





85

$\boxed{1}$ Find the volume of the Solid generated by revolving the region bounded by the Curve $y = x^2$, $y = 2-x$ and $x=0$ for $x \geq 0$ about y-axis. (Q9 page 324)



$$V = \int_{\boxed{0}}^{\boxed{1}} 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix} \begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx$$

$$\int_0^1 2\pi (x)(2-x-x^2)\, dx$$

$$= \frac{5\pi}{6}$$

Intersection point

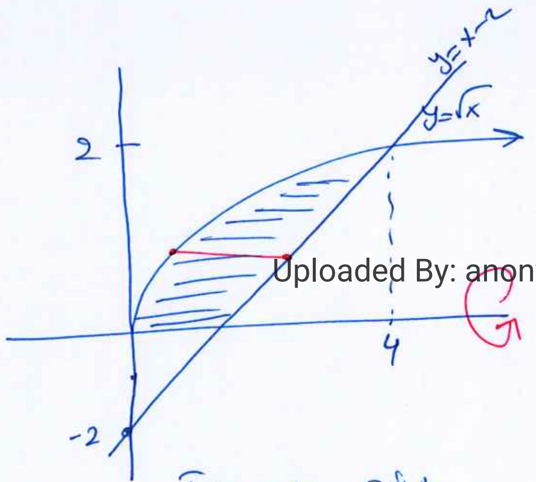
$x^2 = 2-x$

$x^2+x-2=0$

$(x+2)(x-1)$

$\boxed{x=-2}$ reject $\boxed{x=1}$ accept

$\boxed{2}$ Find the volume of the solid generated by revolving the region bounded by the Curve $y=\sqrt{x}$, $y=0$, $y=x-2$ about x-axis (Q21 page 325)



$$V = \int_{\boxed{0}}^{\boxed{2}} 2\pi \begin{pmatrix}\text{Shell}\\ \text{radius}\end{pmatrix} \begin{pmatrix}\text{Shell}\\ \text{length}\end{pmatrix} dy$$

$$= \int_0^2 2\pi (y)(y+2-y^2)\, dy$$

$$= 2\pi \int_0^2 y^2+2y-y^3\, dy$$

$$= \frac{16\pi}{3}.$$

Intersection point

$x-2 = \sqrt{x}$

$x-\sqrt{x}-2=0$

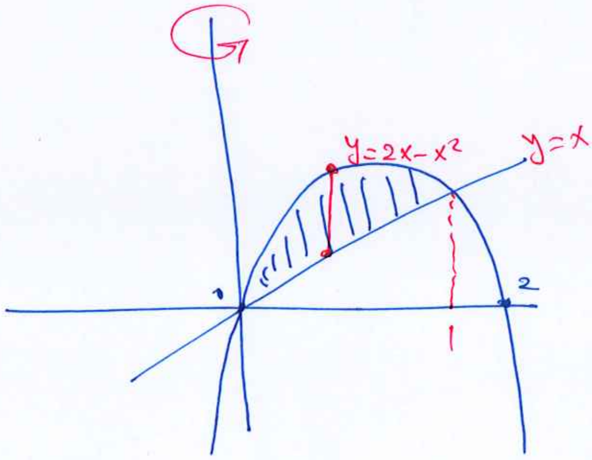
$(\sqrt{x}+1)(\sqrt{x}-2)$

$\boxed{\sqrt{x}=-1}$ reject $\boxed{\sqrt{x}=2} \rightarrow \boxed{x=4}$

$\boxed{3}$ Find the volume of the solid generated by revolving the region bounded by $y = 2x - x^2$, $y = x$ $\boxed{86}$

Case ① about the y-axis

Case ② about the line $x=1$



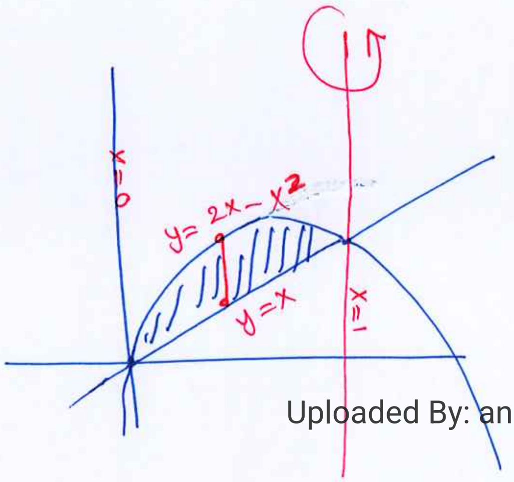
By using Shell Method

$$V = \int_{\boxed{0}}^{\boxed{1}} 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dx$$

$$\int_0^1 2\pi (x)(2x - x^2 - x)\, dx$$

$$2\pi \int_0^1 x(x - x^2)\, dx = \frac{\pi}{6}$$

about $x=1$



$$V = \int_{\boxed{0}}^{\boxed{1}} 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dx$$

$$= \int_0^1 2\pi (1 - x)(2x - x^2 - x)\, dx$$

$$= 2\pi \int_0^1 x - 2x^2 + x^3 \quad dx$$

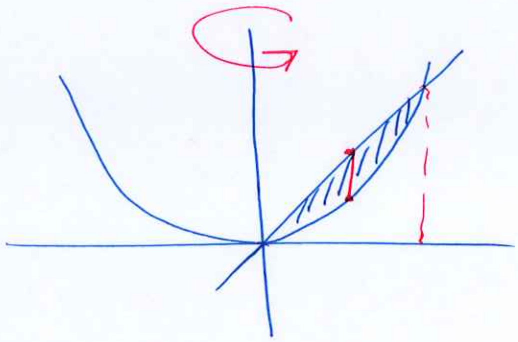
$$= \frac{\pi}{6}.$$

4 Find the volume of the solid generated by revolving the region bounded by $y=x$, $y=x^2$, about y-axis

1 By using Shell Method

2 By using washer ~~#~~ Method

(Q29 Page 325)



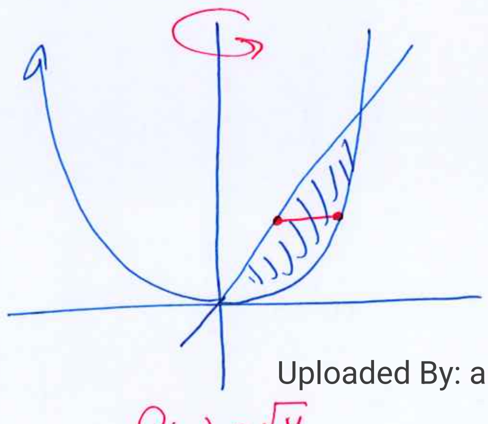
1

$$V=\int_{0}^{1} 2\pi \begin{pmatrix}\text{shell}\\ \text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\ \text{height}\end{pmatrix} dx$$

$$=\int_0^1 2\pi\,(x)\,(x-x^2)\,dx$$

$$=2\pi\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1=\frac{\pi}{6}$$

2



$R(y)=\sqrt{y}$

$r(y)=y$

$$V=\int_{0}^{1} A(y)\,dy$$

$$=\int_0^1 \pi\left[R^2(y)-r^2(y)\right]dy$$

$$=\int_0^1 \pi\left[y-y^2\right]dy$$

$$=\pi\left[\frac{y^2}{2}-\frac{y^3}{3}\right]_0^1=\frac{\pi}{6}.$$