1.5 Elementary Matrices

* **Equivalent Systems:** Given $m\times n$ linear system $\boxed{\underset{m\times n}{A}\,\underset{n\times 1}{x} = \underset{m\times 1}{b}}$ *.

→ If M is $m\times m$ nonsingular matrix, then the linear system $\boxed{MAx = Mb}$ ⊛ is equivalent to * because

⇒ any solution of * is a slo u tion of ⊛

⇒ any solution $\hat{x}$ of ⊛ is a solution of *

because $M^{-1}(MA\hat{x}) = M^{-1}(Mb)$

$$A\hat{x} = b$$

→ To transform the system * to a simpler form that is easier to solve, we multiply * by a sequence of nonsingular matrices $E_k E_{k-1} \cdots E_1$: $E_k E_{k-1} \cdots E_1 A x = E_k E_{k-1} \cdots E_1 b$

to get new system $\boxed{Ux = c}$ ♡, where

- $U = E_k E_{k-1} \cdots E_1 A$ is an upper triangular matrix and
- $c = E_k E_{k-1} \cdots E_1 b$ is the constant vector

⇒ The new transformed system ♡ will be equivalent to * with $M = E_k E_{k-1} \cdots E_1$ is nonsingular (since it's product of nonsingular matrices).

Elementray Matrices are matrices that result by applying exactly one elementray row operation on the identity matrix I.

→ The are three types of elementray matrices crrosponding to the three types of elementary row operations:

Type I An elementary matrix of type I is a matrix obtained by interchanging two rows of the identity I.

Exp¹ $E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an elementray matrix of type I since it was obtained by interchanging the first two raws of I. If A is 3x3 matrix, then

$$E_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A E_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{12} & a_{11} & a_{13} \\ a_{22} & a_{21} & a_{23} \\ a_{32} & a_{31} & a_{33} \end{bmatrix}$$

Type II An elementary matrix of type II is a matrix obtaining by multiplying a row of I by a nonzero constant.

Exp² $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is an elementary matrix of type II.

If A is 3x3 matrix, then

$$E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 5a_{31} & 5a_{32} & 5a_{33} \end{bmatrix}$$

$$A E_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 5a_{13} \\ a_{21} & a_{22} & 5a_{23} \\ a_{31} & a_{32} & 5a_{33} \end{bmatrix}$$

Type III An elementary matrix of type III is a matrix obtained from I by adding a multiple of one row to another row.

Exp³ $E_3 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an elementary matrix of type III.

$$E_3 A = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{11}+5a_{31} & a_{12}+5a_{32} & a_{13}+5a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$A E_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & 5a_{11}+a_{13} \\ a_{21} & a_{22} & 5a_{21}+a_{23} \\ a_{31} & a_{32} & 5a_{31}+a_{33} \end{bmatrix}$$

Th If E is an elementary matrix, then E is nonsingular and E^{-1} is an elementary matrix of the same type.

do the proof

- In Exp^1 $\Rightarrow E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_1^{-1} = E = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. $E_1 E_1^{-1} = I$
- In Exp^2 $\Rightarrow E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \Rightarrow E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$. Hence $E_2 E_2^{-1} = I$
- In Exp^3 $\Rightarrow E_3 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_3^{-1} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Hence $E_3 E_3^{-1} = I$

Def The matrix B is row equivalent to a matrix A if $\exists$ a finite sequence $E_1, E_2, \cdots, E_k$ of elementary matrices s.t

$$B = E_k E_{k-1} \cdots E_1 A$$

Notes 1) If A is row equivalent to B, then B is row equivalent to A.

$$A = E_k E_{k-1} \cdots E_1 B \Leftrightarrow B = E_1^{-1} E_2^{-1} \cdots E_k^{-1} A$$

2) If A is row equivalent to B and B is row equivalent to C, then A is row equivalent to C.

If $A = E_k E_{k-1} \cdots E_1 \underline{\underline{B}}$ and $B = F_m F_{m-1} \cdots F_1 C$ then

$$A = E_k E_{k-1} \cdots E_1 \underline{F_m F_{m-1} \cdots F_1} C$$

Th* Let A be $n \times n$ matrix. The following are equivalent:

a) A is nonsingular b) $\boxed{Ax = 0}$* has only the trivial solution 0

c) A is row equivalent to I.

Proof: a) $\Rightarrow$ b) If A is nonsingular and y is a solution to *

then $y = Iy = (A^{-1}A)y = A^{-1}(Ay) = A^{-1}0 = 0$

b) $\Rightarrow$ c) $Ax = 0$ can be transformed to $Ux = 0$ where U is in row echelon form and U is strictly triangular matrix with diagonal elements all 1 (otherwise, the system will have infinitely many solution).

$\Rightarrow$ Hence I is reduced row echelon form of A.
$\Rightarrow$ A is row equivalent to I.

$\boxed{c} \Rightarrow \boxed{a}$ If A is row equivalent to $I \Rightarrow \exists$ elementary matrices $E_1, E_2, \dots, E_k$ s.t $A = E_k E_{k-1} \cdots E_1 I$
$= E_k E_{k-1} \cdots E_1$

since E_i is invertible $\Rightarrow A^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$ which is invertible too

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Corollary The system $Ax = b$ of n linear equations and n unknowns has a unique solution iff A is nonsingular.

Proof $\Leftarrow$ Let A be nonsingular. if $Ax = b$ then
$A^{-1}Ax = A^{-1}b \Rightarrow x = A^{-1}b$ is the unique solution.

$\Rightarrow$ Assume $Ax = b$ has a unique solution $\hat{x}$ and A is singular (by Contradiction).
since A is singular it follows that $Ax = 0$ has a solution $z \neq 0$. Hence, $y = \hat{x} + z$ is a second solution of $Ax = b$ since

by Th*

$Ay = A(\hat{x} + z) = A\hat{x} + Az = b + 0 = b$ $\times$.
since $Ax = b$ has a unique solution $\hat{x}$.
So A is nonsingular.

Note If A is nonsingular $\Rightarrow$ A is row equivalent to I
$\Rightarrow$ I is row equivalent to A $\Rightarrow$ there exist elementary matrices $E_1, E_2, \dots, E_k$
s.t $E_k E_{k-1} \cdots E_1 A = I$ Hence
$E_k E_{k-1} \cdots E_1 = A^{-1}$

- This provides a way to find A^{-1}
- We augment A by I: $[A \mid I]$ then perform elementry row operations that transform A to I: $[I \mid A^{-1}]$.

Exp Let $A = \begin{bmatrix} -1 & -3 & -3 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{bmatrix}$. Find A^{-1}

$$\left[\begin{array}{ccc|ccc} -1 & -3 & -3 & 1 & 0 & 0 \\ 2 & 6 & 1 & 0 & 1 & 0 \\ 3 & 8 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3+3R_1]{R_2+2R_1} \left[\begin{array}{ccc|ccc} -1 & -3 & -3 & 1 & 0 & 0 \\ 0 & 0 & -5 & 2 & 1 & 0 \\ 0 & -1 & -6 & 3 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} -1 & -3 & -3 & 1 & 0 & 0 \\ 0 & -1 & -6 & 3 & 0 & 1 \\ 0 & 0 & -5 & 2 & 1 & 0 \end{array}\right] \xrightarrow[R_1 - \frac{3}{5}R_3]{R_2 - \frac{6}{5}R_3} \left[\begin{array}{ccc|ccc} -1 & -3 & 0 & -\frac{1}{5} & -\frac{3}{5} & 0 \\ 0 & -1 & 0 & \frac{3}{5} & -\frac{6}{5} & 1 \\ 0 & 0 & -5 & 2 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - 3R_2} \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & -2 & 3 & -3 \\ 0 & -1 & 0 & \frac{3}{5} & -\frac{6}{5} & 1 \\ 0 & 0 & -5 & 2 & 1 & 0 \end{array}\right] \Longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -3 & 3 \\ 0 & 1 & 0 & -\frac{3}{5} & \frac{6}{5} & -1 \\ 0 & 0 & 1 & -\frac{2}{5} & -\frac{1}{5} & 0 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 2 & -3 & 3 \\ -\frac{3}{5} & \frac{6}{5} & -1 \\ -\frac{2}{5} & -\frac{1}{5} & 0 \end{bmatrix}$$

Exp solve the system

$$-x_1 - 3x_2 - 3x_3 = 5$$
$$2x_1 + 6x_2 + x_3 = 5$$
$$3x_1 + 8x_2 + 3x_3 = 0$$

$$Ax = b \Rightarrow x = A^{-1}b$$

$$\begin{bmatrix} -1 & -3 & -3 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & -3 & -3 \\ 2 & 6 & 1 \\ 3 & 8 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -3 & 3 \\ -\frac{3}{5} & \frac{6}{5} & -1 \\ -\frac{2}{5} & -\frac{1}{5} & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \\ -3 \end{bmatrix}$$

Diagonal and Triangular Matrices

* The $n \times n$ matrix is upper triangular if $a_{ij} = 0$ for $i > j$
* The $n \times n$ matrix is lower triangular if $a_{ij} = 0$ for $i < j$
* The $n \times n$ matrix is triangular if it is either upper triangular or lower triangular

Exp $\underbrace{\begin{bmatrix} 3 & 1 & 2 \\ 0 & 5 & 3 \\ 0 & 0 & -1 \end{bmatrix}}_{\text{upper}}$ and $\underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 4 & 0 & 0 \\ 1 & 3 & 4 \end{bmatrix}}_{\text{lower}}$ are triangular.

Note that ① triangular matrix may have zero on the diagonal.

② For the linear system $Ax = b$ to be in strict triangular form, the coefficient matrix A must be in upper triangular with nonzero diagonal entries.

* The $n \times n$ matrix A is diagonal if $a_{ij} = 0$ for $i \neq j$

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ are all diagonals.}$$

* The diagonal matrix is both upper and lower triangular.

Triangular Factorization

Let $A = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix}$ using row operation III only

- We can find a unit lower triangular matrix L and an upper triangular matrix U s.t $LU = A$:

- $$\begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} \quad \begin{matrix} l_{21} = \frac{1}{2} \\ l_{31} = \frac{4}{2} = 2 \end{matrix} \quad \xrightarrow[R_3 - \boxed{2} R_1]{R_2 - \boxed{\frac{1}{2}} R_1} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & -9 & 5 \end{bmatrix} \quad l_{32} = \frac{-9}{3} = -3 \quad \xrightarrow[R_3 - \boxed{-3} R_2]{} \underbrace{\begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix}}_{U}$$

- $$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}$$ "L is unit lower triangular since it's diagonal is 1"

- $$LU = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 2 \\ 1 & 5 & 2 \\ 4 & -1 & 9 \end{bmatrix} = A$$

- The factorization of the matrix A into a product of a unit lower triangular matrix L and strictly upper triangular matrix U is called the LU factorization.

- In this factorization, we applied three row operations to the matrix A. Hence, we have three elementary matrices E_1, E_2, E_3: of type III :

- $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l_{31} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \Rightarrow E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$

$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -l_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \Rightarrow E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$

- Note that $\boxed{E_3 E_2 E_1 A = U}$

since the elementary matrices are nonsingular we can multiply by their inverses.

- Hence,

$$A = (E_3 E_2 E_1)^{-1} U$$

$$\boxed{A = E_1^{-1} E_2^{-1} E_3^{-1} U} = LU \quad \text{where } \boxed{L = E_1^{-1} E_2^{-1} E_3^{-1}}$$

Notes: If A is $n \times n$ matrix that can be reduced to strict upper triangular form using only row operation III, then A has LU factorization.
