

3.3 Differentiation Rules

53

① Derivative of a constant function

If $f(x) = c$, then $\frac{df}{dx} = 0$

Proof: $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = 0$

② Power Rule

If $f(x) = x^n$, then $f'(x) = n x^{n-1}$ $n \in \mathbb{R}$
for all x , where x^n and x^{n-1} are defined.

Example ① Let $f(x) = \sqrt{3} \Rightarrow f'(x) = 0$

② $f(x) = x^3 \Rightarrow f'(x) = 3x^2$

③ $f(x) = \frac{1}{x^4} \Rightarrow f'(x) = -4x^{-5} = \frac{-4}{x^5}$

④ $g(x) = \sqrt{x^2 + \pi} \Rightarrow g(x) = (x^2 + \pi)^{\frac{1}{2}} = x^{1 + \frac{\pi}{2}}$

$\Rightarrow g'(x) = (1 + \frac{\pi}{2}) x^{\frac{\pi}{2}} = (1 + \frac{\pi}{2}) \sqrt{x^\pi}$

STUDENTS HUB.com $h(x) = x^{\sqrt{3}} \Rightarrow h'(x) = \sqrt{3} x^{\sqrt{3}-1}$

Uploaded By: Malak Obaid

Let $f(x) = cu(x)$

③ where $u(x)$ is a differentiable function of x and c is a constant, then

$$f'(x) = \frac{d}{dx} (cu) = c \frac{du}{dx} = c u'(x)$$

Proof: $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c u(x+h) - c u(x)}{h}$ (54)

$$= c \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} = c u'(x)$$

Example: ① If $f(x) = 5x^2$, then $f'(x) = (5 \times 2)x = 10x$
 ② If $f(x) = -\sqrt{3}x^3$, then $f'(x) = -3\sqrt{3}x^2$

④ Derivative sum Rule:

If $f(x) = u(x) + v(x)$ where u and v are differentiable functions of x , then f is differentiable at every point where u and v are both differentiable

$$f'(x) = u'(x) + v'(x)$$

Proof: $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{u(x+h) + v(x+h) - u(x) - v(x)}{h}$

$$= \lim_{h \rightarrow 0} \left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h} \right]$$

$$= u'(x) + v'(x)$$

Example ① Let $f(x) = x^3 - \frac{3}{2}x^2 - 7x + 3$.

$$f'(x) = 3x^2 - 3x - 7$$

② Does the curve $g(x) = x^2 + 1$ have any horizontal tangents?

Horizontal tangents $\Rightarrow g'(x) = 0 = 2x \Rightarrow \boxed{x = 0}$

$$(0, g(0)) = (0, 1)$$

⑤ Derivative Product Rule:

Let $f(x) = g(x) p(x)$, where g and p are differentiable at x , then f is differentiable at x :

$$f'(x) = g(x) p'(x) + p(x) g'(x)$$

Example: Find the derivative of $y = (x^2 - 1)(x^3 + 3)$

$$y' = (x^2 - 1)(3x^2) + (x^3 + 3)(2x)$$

$$= 3x^4 - 3x^2 + 2x^4 + 6x$$

$$= 6x^4 - 3x^2 + 6x$$

Proof: $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{g(x+h)p(x+h) - g(x)p(x)}{h}$

$$= \lim_{h \rightarrow 0} \frac{g(x+h)p(x+h) - g(x+h)p(x) + g(x+h)p(x) - g(x)p(x)}{h}$$

$$= \lim_{h \rightarrow 0} \left[g(x+h) \frac{p(x+h) - p(x)}{h} + p(x) \frac{g(x+h) - g(x)}{h} \right]$$

$$= \lim_{h \rightarrow 0} g(x+h) \lim_{h \rightarrow 0} \frac{p(x+h) - p(x)}{h} + p(x) \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}$$

$$= g(x) p'(x) + p(x) g'(x)$$

⑥ Derivative Quotient Rule:

Let $f(x) = \frac{g(x)}{p(x)}$, $p(x) \neq 0$, p and g are differentiable at x , then f is differentiable at x :

$$f'(x) = \frac{p(x) g'(x) - g(x) p'(x)}{p^2(x)}$$

(56)

Example: Find the derivative of $y = \frac{3x-4}{x^2+1}$

$$y' = \frac{(x^2+1)(3) - (3x-4)(2x)}{(x^2+1)^2}$$

$$= \frac{3x^2+3 - 6x^2+8x}{(x^2+1)^2} = \frac{3+8x-3x^2}{(x^2+1)^2}$$

Proof: $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{g(x+h)}{p(x+h)} - \frac{g(x)}{p(x)}}{h}$

$$= \lim_{h \rightarrow 0} \frac{p(x)g(x+h) - g(x)p(x+h)}{h p(x) p(x+h)}$$

$$= \lim_{h \rightarrow 0} \frac{p(x)g(x+h) - p(x)g(x) + p(x)g(x) - g(x)p(x+h)}{h p(x) p(x+h)}$$

$$= \lim_{h \rightarrow 0} \frac{p(x) \frac{g(x+h) - g(x)}{h} - g(x) \frac{p(x+h) - p(x)}{h}}{p(x) p(x+h)}$$

$$= \frac{p(x)g'(x) - g(x)p'(x)}{p^2(x)}$$

(6) **Second and Higher - Order Derivatives:** Let $y = f(x)$

$$y' = f'(x) = \frac{dy}{dx} \quad \text{1st derivative "y prime"}$$

$$y'' = f''(x) = \frac{d^2y}{dx^2} \quad \text{2nd derivative "y double prime"}$$

$$y''' = f'''(x) = \frac{d^3y}{dx^3} \quad \text{3rd derivative "y triple prime"}$$

$$\vdots$$

$$y^{(n)} = f^{(n)}(x) = \frac{d^ny}{dx^n} \quad \text{nth derivative "y super n"}$$

Assuming f has
all derivative

Uploaded By: Malak Obaid

n is positive
integer

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

Example: Find the derivatives of all orders of the function $y = x^3 - 3x^2 + 2$

first derivative $y' = 3x^2 - 6x$

Second derivative $y'' = 6x$

Third derivative $y''' = 6$

Fourth derivative $y^{(4)} = 0$

The fifth and later derivatives are zero.

Proof of (2) $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x}$

$= \lim_{z \rightarrow x} \frac{z^n - x^n}{z - x}$ ← انما استعملت وزع

$= \lim_{z \rightarrow x} \frac{(z-x) \left(z^{n-1} + z^{n-2}x + \dots + z^{n-2}x^{n-2} + x^{n-1} \right)}{(z-x)}$

$= \lim_{z \rightarrow x} \left(z^{n-1} + z^{n-2}x + \dots + z^{n-2}x^{n-2} + x^{n-1} \right)$

$= n x^{n-1}$