Control System Engineering

Learning Unit 10

Frequency Response Techniques

Learning Unit 9 – Frequency Response Techniques

- Define and plot the frequency response of a system.
- Plot asymptotic approximations to the frequency response of a system.
- Find gain and phase margins. Also study the stability based on bode plots.
- Find the closed-loop time response parameters of peak time, settling time, and percentage overshoot given the open-loop frequency response.

Learning Unit 10 – Frequency Response Techniques

- **The Concept of Frequency Response**

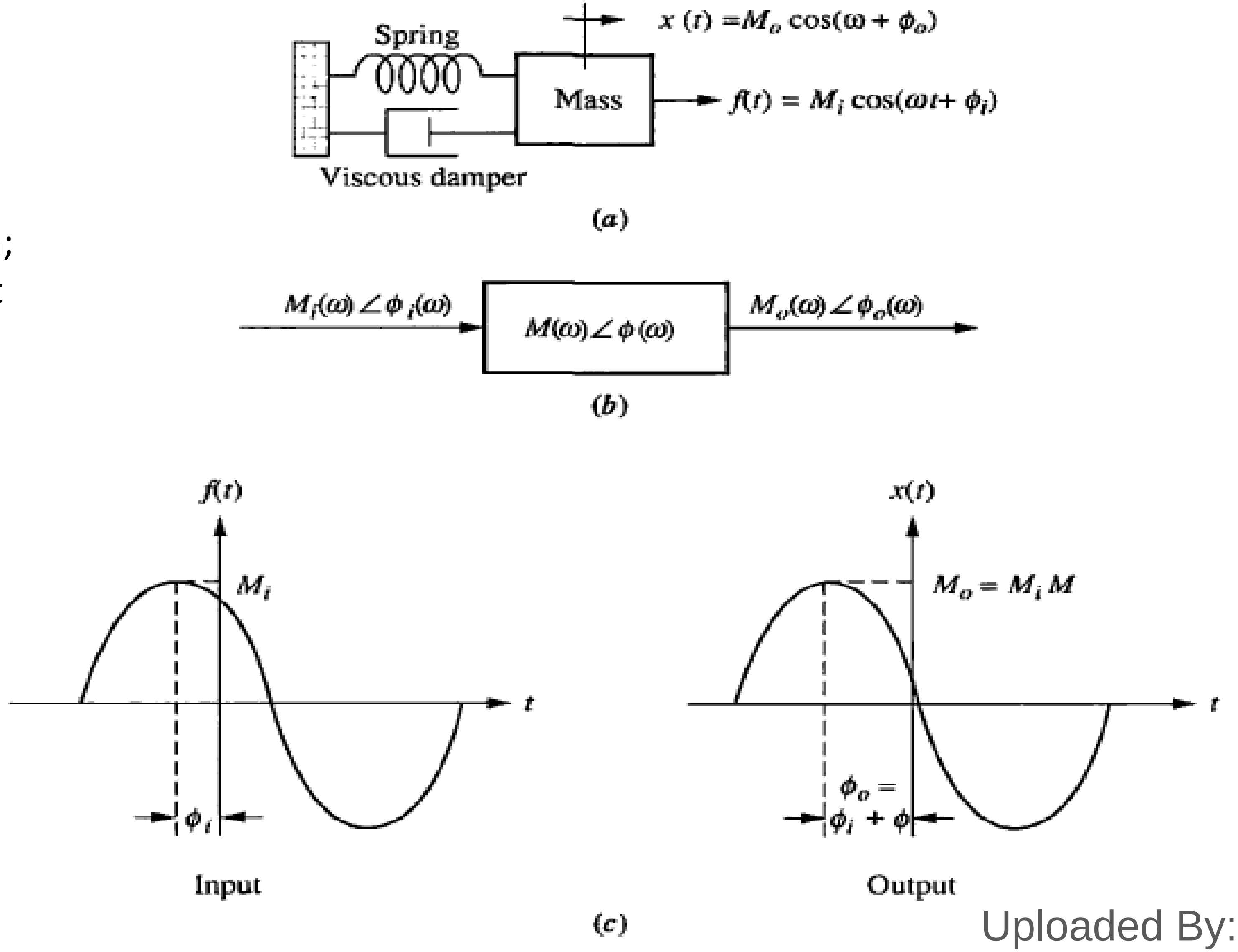
In the steady state, sinusoidal inputs to a linear system generate sinusoidal responses of the same frequency.

These responses are of the same frequency as the input, they differ in amplitude and phase angle from the input.

Sinusoids can be represented as complex numbers called phasors. $M_1 \cos(\omega t + \phi_1) = M_1 \angle \phi_1$

Learning Unit 10 – Frequency Response Techniques

a. system;
b. transfer function;
c. input and output waveforms



Learning Unit 10 – Frequency Response Techniques

- Steady-state output sinusoid is

$$M_o(\omega)\angle\phi_o(\omega) = M_i(\omega)M(\omega)\angle[\phi_i(\omega) + \phi(\omega)]$$

- The system function is given by

$$M(\omega) = \frac{M_o(\omega)}{M_i(\omega)} \qquad \phi(\omega) = \phi_o(\omega) - \phi_i(\omega)$$

- We call $M(\omega)$ the *magnitude frequency response* and $\phi(\omega)$ the *phase frequency response*.

- The combination of the magnitude and phase frequency responses is called the *frequency response* and is $M(\omega)\angle\phi(\omega)$

Asymptotic Approximations: Bode Plots

- The log-magnitude and phase frequency response curves as functions of log ω are called Bode plots or Bode diagrams.
- Sketching Bode plots can be simplified because they can be approximated as a sequence of straight lines.
- Straight-line approximations simplify the evaluation of the magnitude and phase frequency response.

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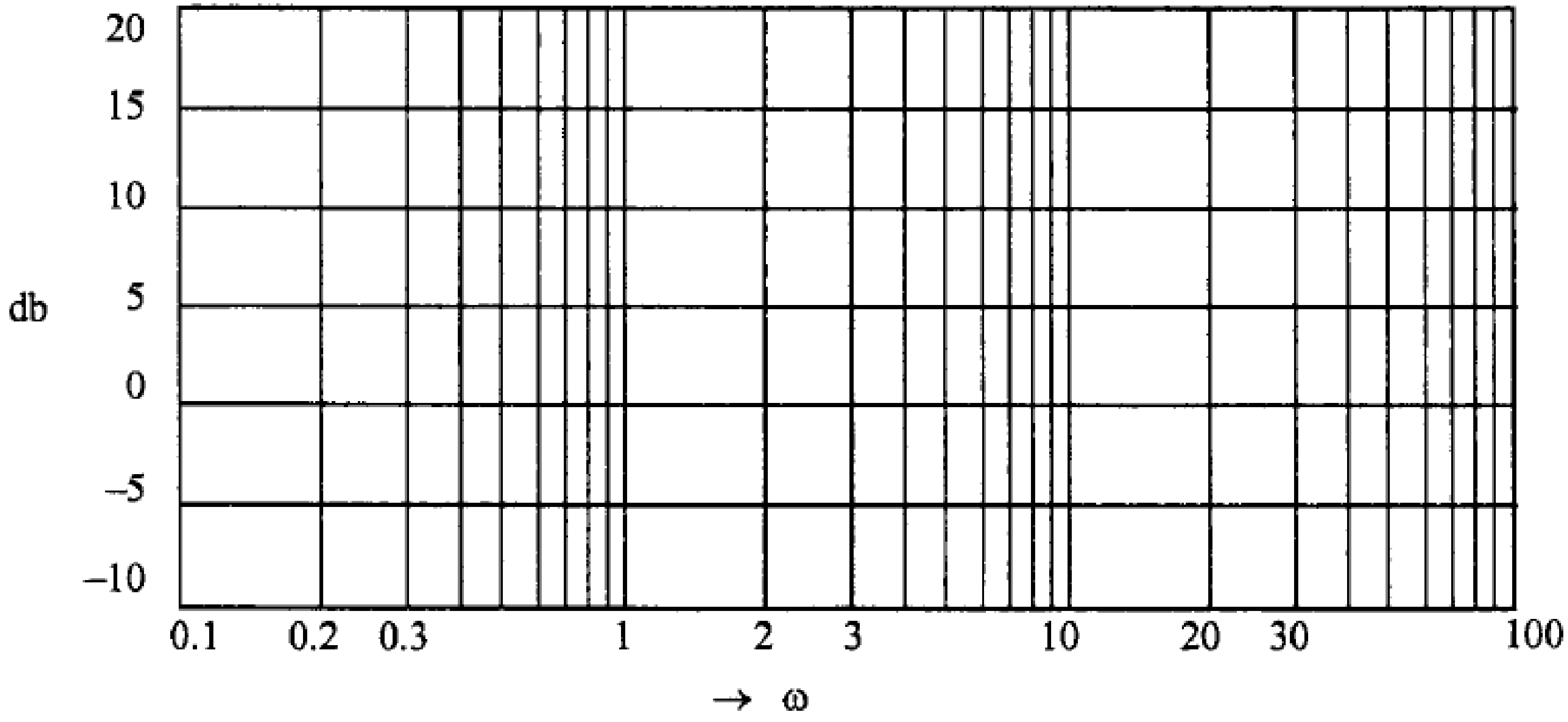

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- Consider the following transfer function

$$G(s) = \frac{K(s+z_1)(s+z_2)\dots(s+z_k)}{s^m(s+p_1)(s+p_2)\dots(s+p_n)}$$

- The magnitude frequency response is the product of the magnitude frequency responses of each term, or

$$|G(j\omega)| = \left(\frac{K|s+z_1||s+z_2|\dots|s+z_k|}{|s^m||s+p_1||s+p_2|\dots|s+p_k|}\right)\Bigg|_{s=j\omega}$$

$$|G(j\omega)| = \frac{K|j\omega+z_1||j\omega+z_2|\dots|j\omega+z_k|}{|(j\omega)^m||j\omega+p_1||j\omega+p_2|\dots|j\omega+p_k|}$$

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Converting the magnitude response into dB, we obtain

$$20\log|G(j\omega)| = 20\log(K) + 20\log|j\omega + z_1| + 20\log|j\omega + z_2| + \cdots + 20\log|j\omega + z_k| - 20\log|(j\omega)^m| - 20\log|j\omega + p_1| - 20\log|j\omega + p_2| - \cdots - 20\log|j\omega + p_k|$$

Phase frequency response is the *sum* of the phase frequency response curves of the zero terms minus the *sum* of the phase frequency response curves of the pole terms.

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$G(s) = s + a$

$G(j\omega) = j\omega + a$

$G(j\omega) = j\omega + a = a(j\frac{\omega}{a} + 1)$

Low frequency $\omega \longrightarrow 0$

$G(j\omega) \approx a$

-The magnitude response in dB is $20 \log M \ = \ 20 \log a$

- The angle approaches zero

High frequency $\omega \gg a$

$G(j\omega) \approx a\left(\frac{j\omega}{a}\right) = \omega\angle 90°$

The magnitude response in dB is $20 \log M = \ 20 \log a + 20 \log\left(\frac{\omega}{a}\right)$ $=20 \log(\omega)$

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- If we plot dB, $20\,log(M)$, against $log(\omega)$, $20\,log(\omega)$ becomes a straight line: *y=20x.*

- where *y* = $20\,log\,(M)$, and $x = log\,(\omega)$. The line has a slope of 20 when plotted as dB vs. $log(\omega)$

- We call the straight-line approximations *asymptotes.* The low-frequency approximation is called the *low-frequency asymptote,* and the high-frequency approximation is called the *high-frequency asymptote.* The frequency, (a), is called the *break frequency* because it is the break between the low- and the high-frequency asymptotes.

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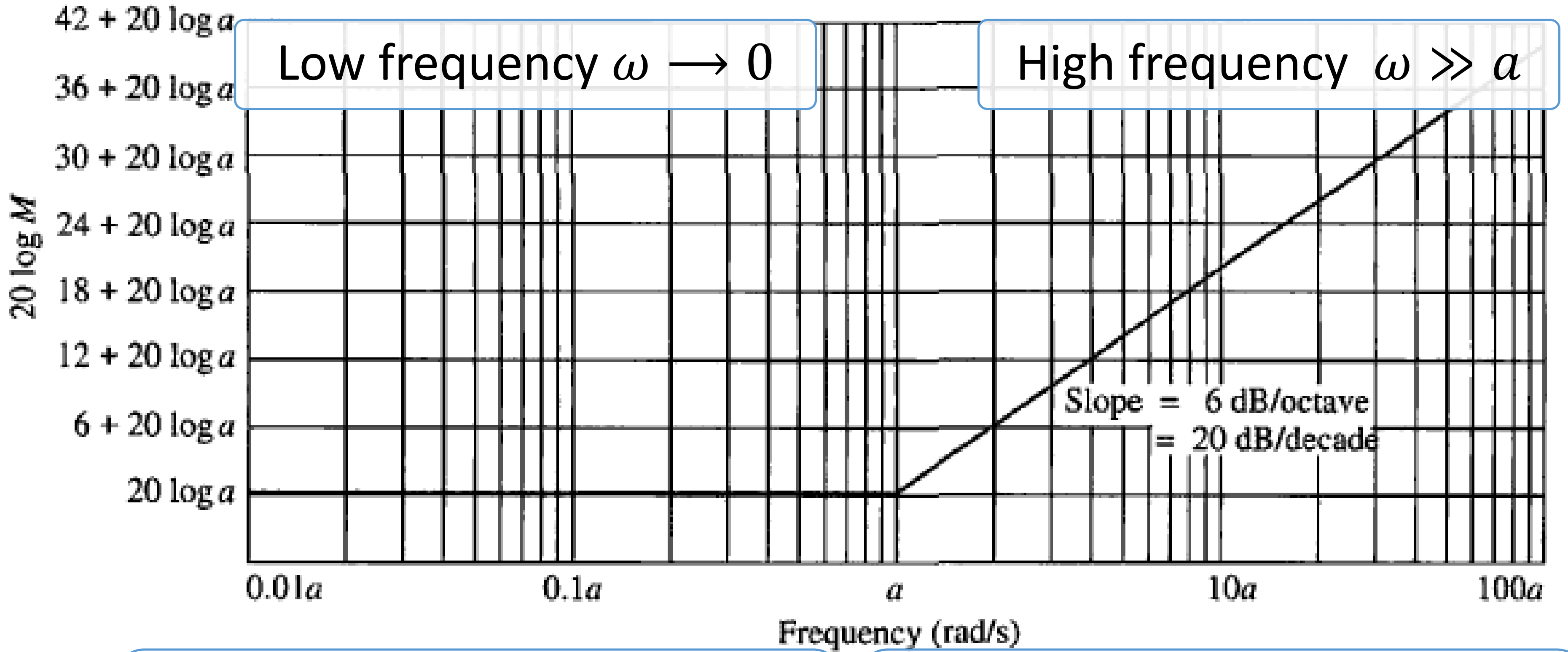

Low frequency $\omega \longrightarrow 0$

High frequency $\omega \gg a$

-The magnitude response in dB is $20 \log M = 20 \log a$

- The angle approaches zero

The magnitude response in dB is $20 \log M = 20 \log a + 20 \log \left(\frac{\omega}{a}\right)$ $=20 \log(\omega)$

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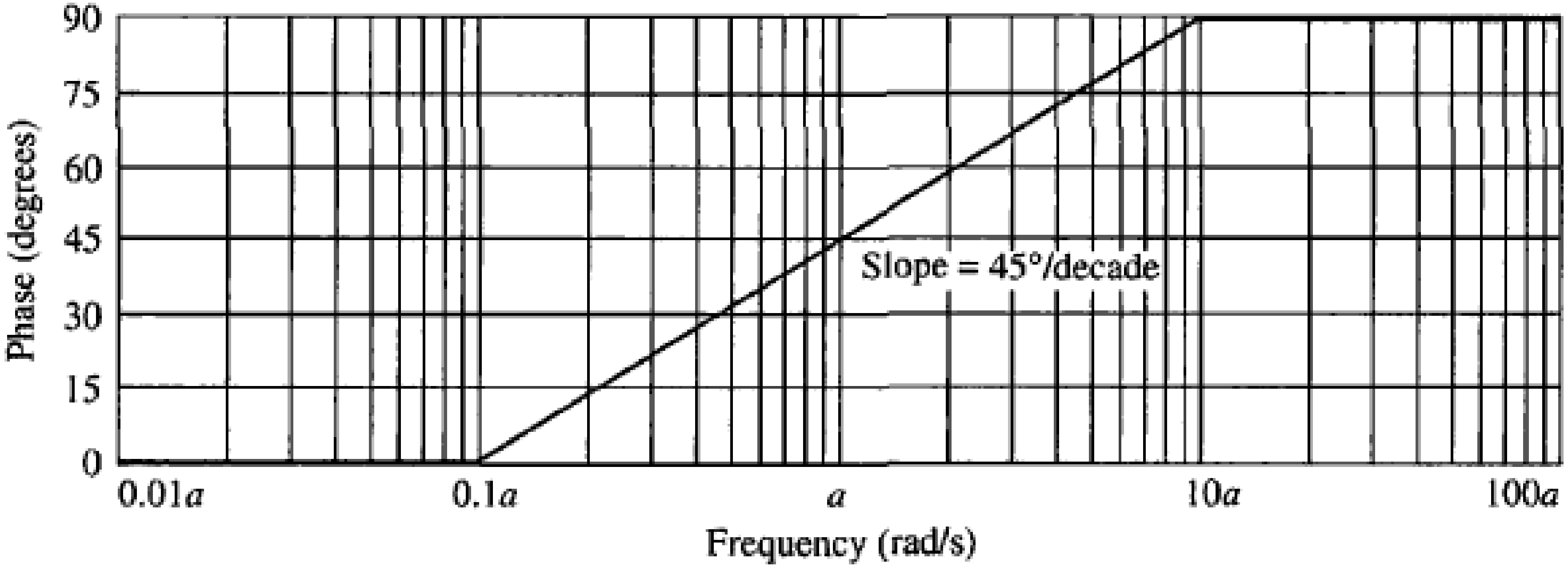

Low frequency $\omega \longrightarrow 0$	High frequency $\omega \gg a$
The angle approaches zero	$G(j\omega) \approx a\left(\frac{j\omega}{a}\right) = \omega\angle 90°$

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Bode Plots for $G(s) = \frac{1}{s+a}$, $G(j\omega) = \frac{1}{j\omega+a} = \frac{1}{a(\frac{j\omega}{a}+1)}$

Magnitude M (dB) :

$$20\log|G(j\omega)| = 20\log|1| - 20\log|a| - 20\log\sqrt{\left(\frac{\omega}{a}\right)^2 + 1}$$

$$20\log|G(j\omega)| = -20\log|a| - 20\log\sqrt{\left(\frac{\omega}{a}\right)^2 + 1}$$

Angle :

$$\varphi = tan^{-1}\left(\frac{0}{1}\right) - tan^{-1}\left(\frac{0}{a}\right) - tan^{-1}\left(\frac{\omega}{a}\right)$$

$$= -tan^{-1}\left(\frac{\omega}{a}\right)$$

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$$G(s) = \frac{1}{s+a}$$

Magnitude M (dB) :

$$20\log|G(j\omega)|$$

$$= -20\log|a| - 20\log\sqrt{\left(\frac{\omega}{a}\right)^2 + 1}$$

Angle :

$$\varphi = -tan^{-1}\left(\frac{\omega}{a}\right)$$

Low frequency

$\omega \to 0$

Magnitude:$-20\log|a|$

Angle:$\varphi = -tan^{-1}\left(\frac{0}{a}\right)$=0

High frequency

$\omega \gg a, \quad \frac{\omega}{a} \gg 1, (\frac{\omega}{a})^2 + 1 \approx (\frac{\omega}{a})^2$

Magnitude =$-20\log(a) - 20\log\frac{\omega}{a}$=$-20\log|\omega|$

Angle=$-tan^{-1}(\infty) = -\frac{\pi}{2}$

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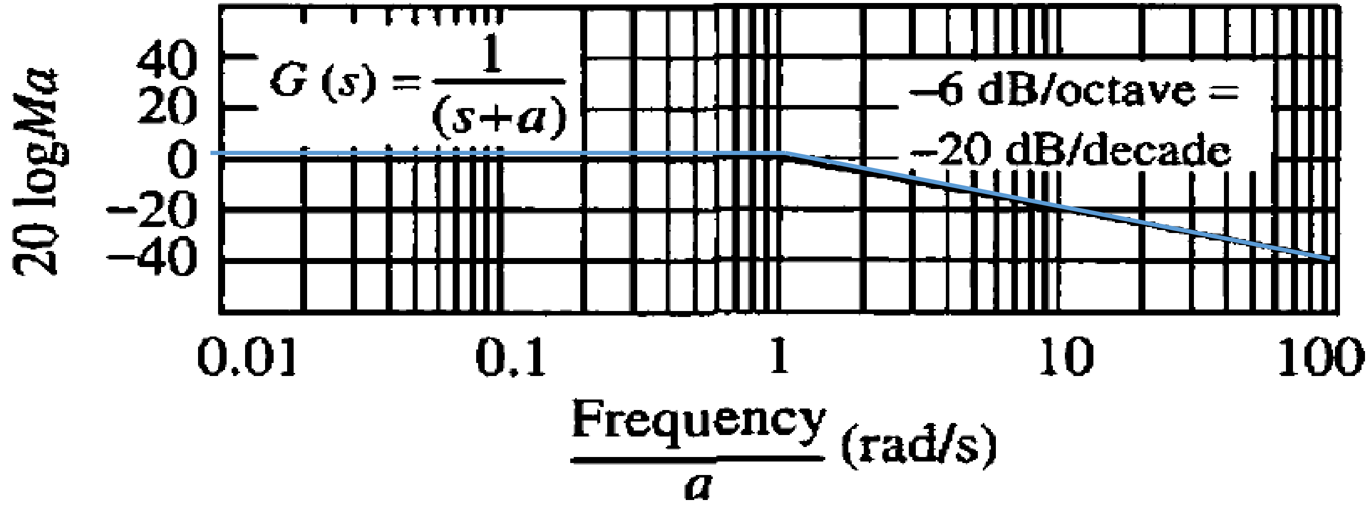

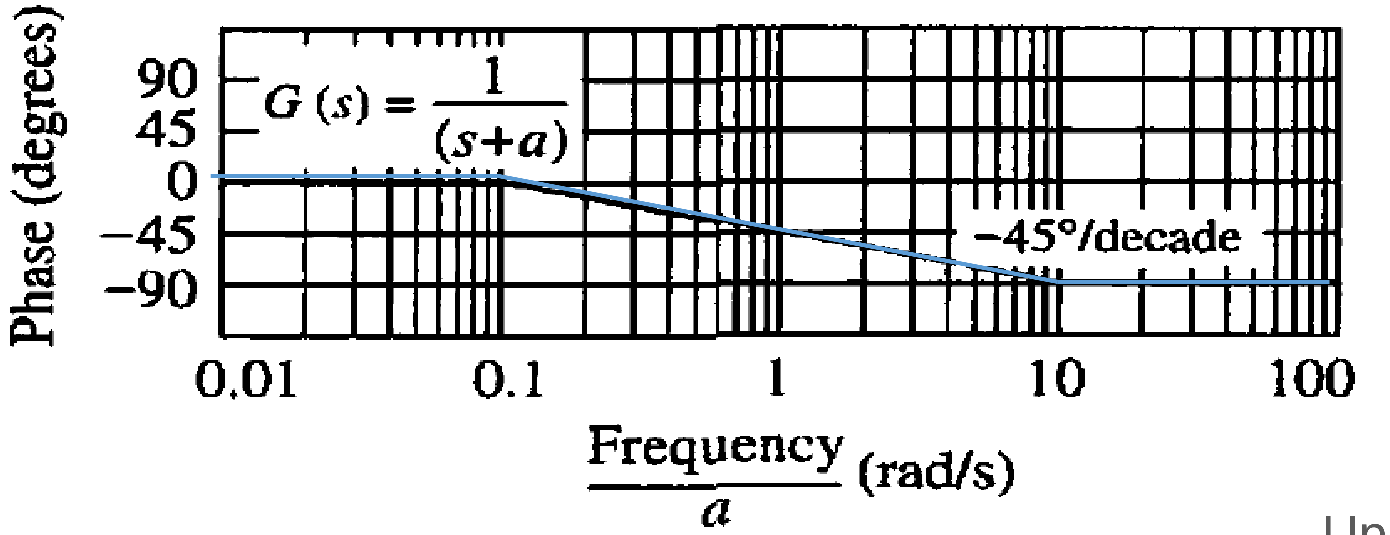

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Bode Plots for $G(s) = s$, $G(j\omega) = j\omega$

Magnitude M (dB) :

$20\log|G(j\omega)| = 20\log|\omega|$

Angle :

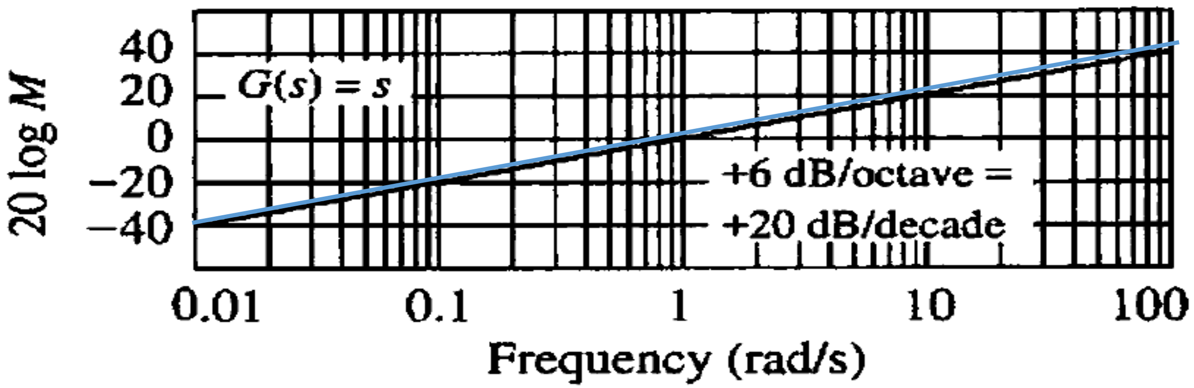
$\varphi = tan^{-1}\left(\frac{\omega}{0}\right) = tan^{-1}(\infty) = \frac{\pi}{2}$

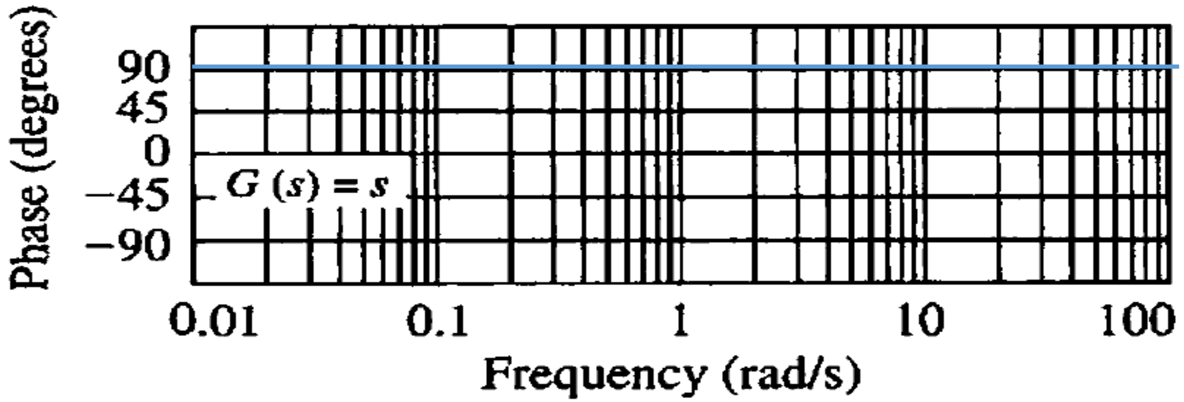
Learning Unit 10 – Frequency Response Techniques

Bode Plots for $G(s) = s$, $G(j\omega) = j\omega$

Magnitude M (dB) : $20\log|G(j\omega)| = 20\log|\omega|$

Angle : $\varphi = tan^{-1}\left(\frac{\omega}{0}\right)$=$tan^{-1}(\infty) = \frac{\pi}{2}$





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Bode Plots for $G(s) = 1/s$, $G(j\omega) = 1/j\omega$

Magnitude M (dB) :

$$20\log|G(j\omega)| = 20\log|1/\omega| = -20\log|\omega|$$

Angle :

$$\varphi = tan^{-1}\left(\frac{0}{1}\right) - tan^{-1}\left(\frac{\omega}{0}\right) = -tan^{-1}(\infty) = -\frac{\pi}{2}$$

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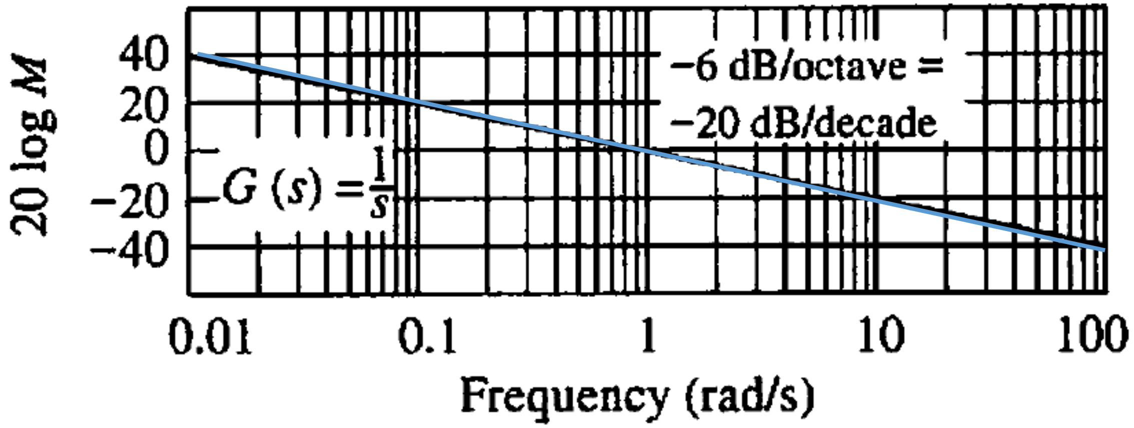
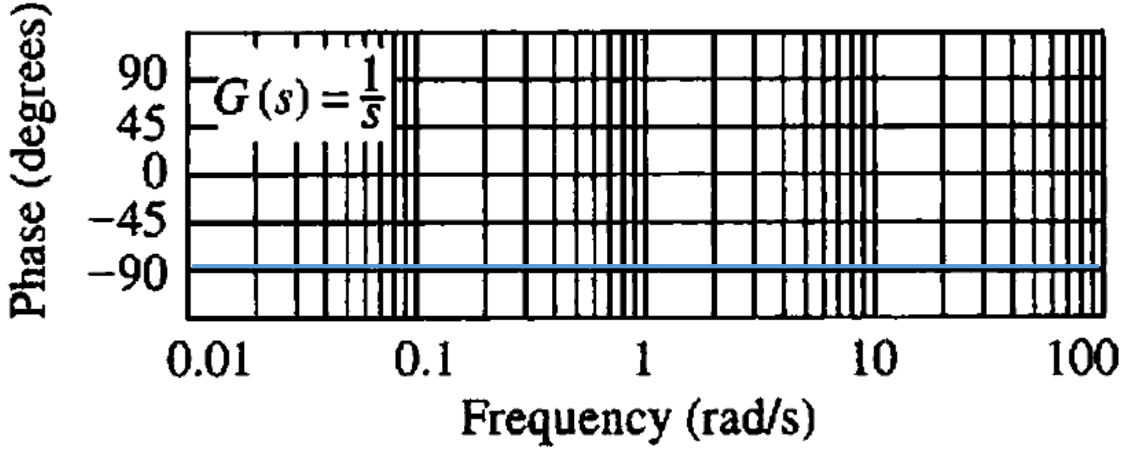
Bode Plots for $G(s) = 1/s$, $G(j\omega) = 1/j\omega$

Magnitude M (dB) : $20\log|G(j\omega)| = -20\log|\omega|$, **Angle :** $\varphi = -\frac{\pi}{2}$





Learning Unit 10 – Frequency Response Techniques

- **Bode Plots for $\boldsymbol{G(s) = s^2 + 2\zeta\omega_n s + \omega_n{}^2}$**

$$G(j\omega) = (j\omega)^2 + 2j\zeta\omega_n\omega + \omega_n{}^2$$

$$= \omega_n{}^2\left(-\left(\frac{\omega}{\omega_n}\right)^2 + 2j\zeta\frac{\omega}{\omega_n} + 1\right)$$

<u>Magnitude M (dB) :</u>

$$20\log|G(s)| = 20\log\omega_n{}^2 + 20\log\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}$$

<u>Angle :</u> $\varphi = tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)$

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M (dB) $=20\log {\omega_n}^2 + 20\log\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2}$, $\varphi = tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)$

$$\boldsymbol{G(s) = s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

$$\boldsymbol{G(j\omega) = -\omega^2 + 2j\zeta\omega_n\omega + {\omega_n}^2}$$

Low frequency : $\boldsymbol{\omega \ll \omega_n, \omega \to 0}$.

Magnitude (dB)$=20\log {\omega_n}^2$=40 log ω_n

$\varphi = tan^{-1}(0) = 0°$

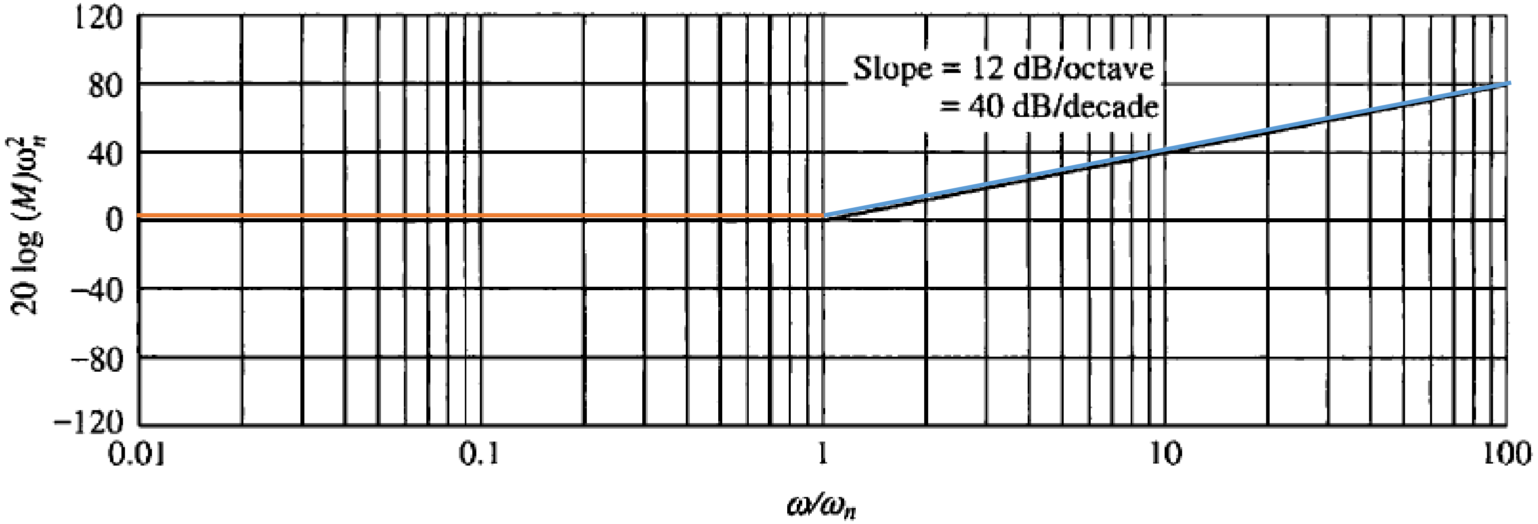
High Frequency : : $\boldsymbol{\omega \gg \omega_n,\ \frac{\omega}{\omega_n} \gg 1, \omega \to \infty}$,

M (dB)$=20\log {\omega_n}^2 + 20\log\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\zeta\frac{\omega}{\omega_n}\right)^2} \approx$

$\approx 20\log {\omega_n}^2$ +20 log $\left(\frac{\omega}{\omega_n}\right)^2$=**40 log $\boldsymbol{\omega}$**

$\boldsymbol{G(j\omega) \approx -\omega^2 = \omega^2 \angle 180°}$

Learning Unit 10 – Frequency Response



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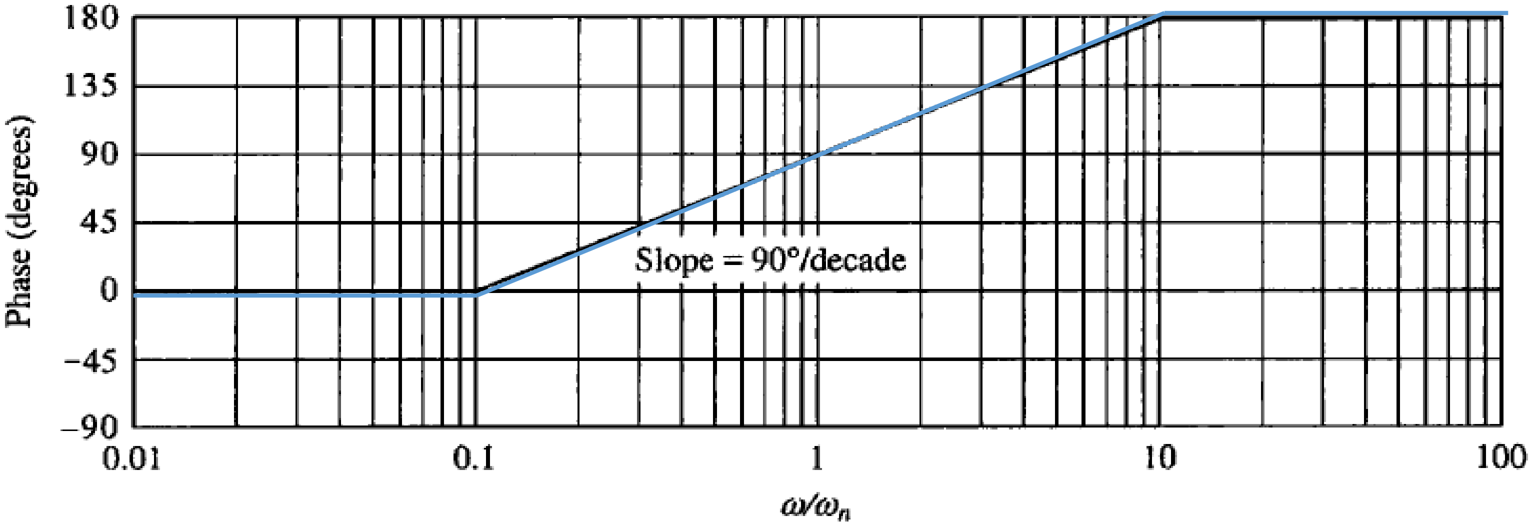

(b)

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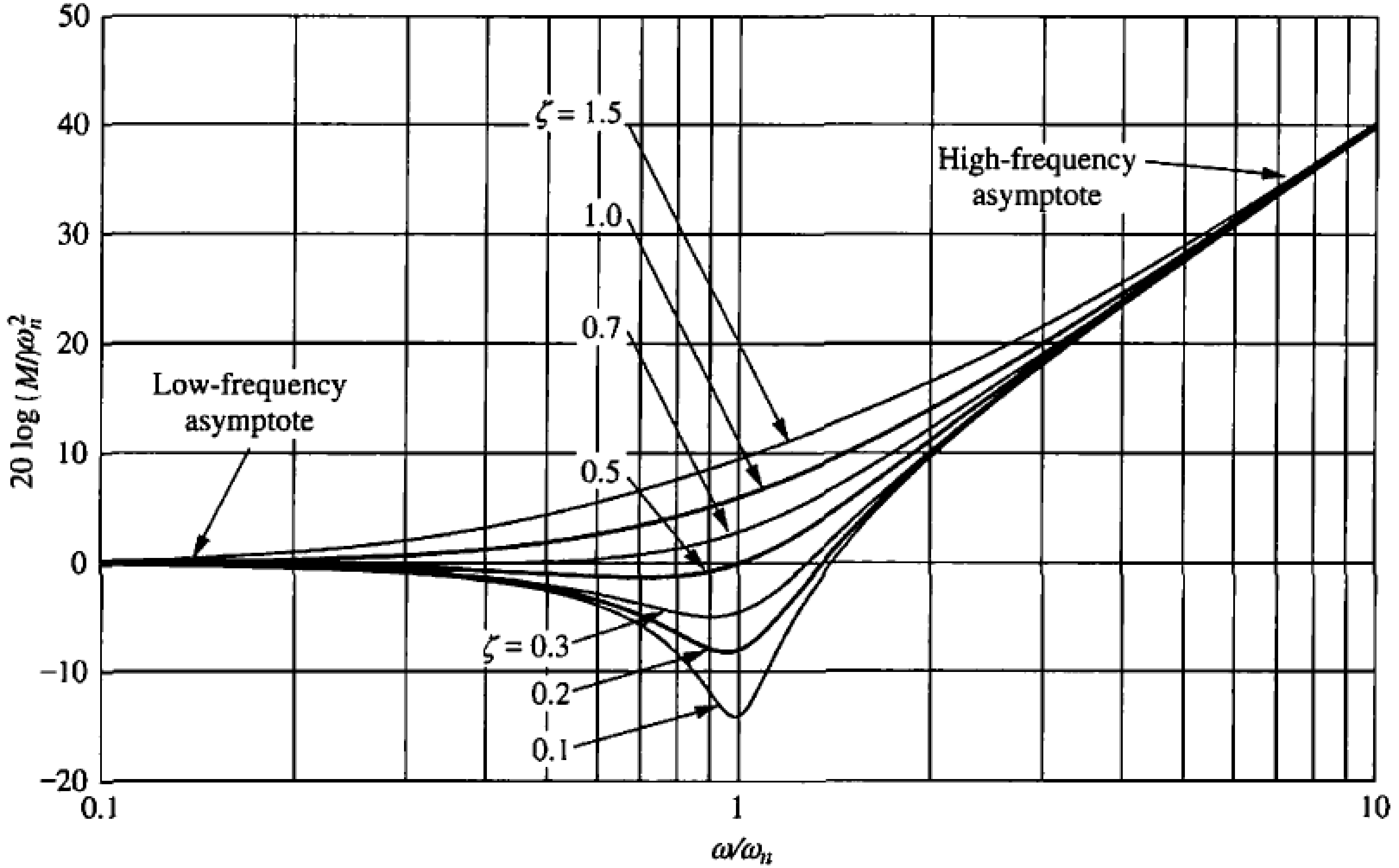

Normalized and scaled log-magnitude response for $G(s) = s^2 + 2\zeta\omega_n s + \omega_n{}^2$

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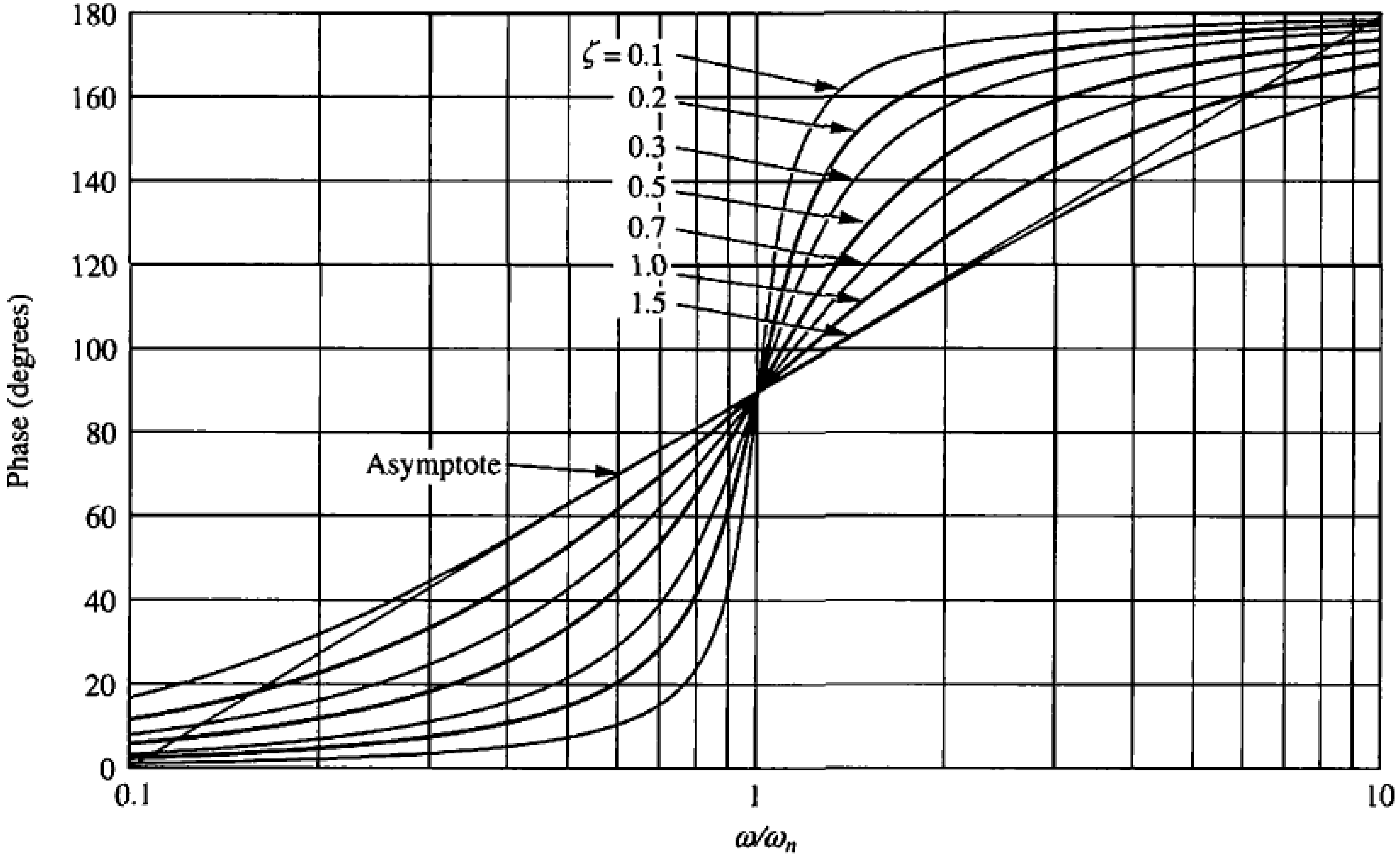


Scaled phase response for $G(s) = s^2 + 2\zeta\omega_n s + {\omega_n}^2$

TABLE 10.4 Data for normalized and scaled log-magnitude and phase plots for $(s^2 + 2\zeta\omega_n s + \omega_n^2)$. Mag $= 20 \log(M/\omega_n^2)$

Freq. $\frac{\omega}{\omega_n}$	Mag (dB) $\zeta = 0.1$	Phase (deg) $\zeta = 0.1$	Mag (dB) $\zeta = 0.2$	Phase (deg) $\zeta = 0.2$	Mag (dB) $\zeta = 0.3$	Phase (deg) $\zeta = 0.3$
0.10	−0.09	1.16	−0.08	2.31	−0.07	3.47
0.20	−0.35	2.39	−0.32	4.76	−0.29	7.13
0.30	−0.80	3.77	−0.74	7.51	−0.65	11.19
0.40	−1.48	5.44	−1.36	10.78	−1.17	15.95
0.50	−2.42	7.59	−2.20	14.93	−1.85	21.80
0.60	−3.73	10.62	−3.30	20.56	−2.68	29.36
0.70	−5.53	15.35	−4.70	28.77	−3.60	39.47
0.80	−8.09	23.96	−6.35	41.63	−4.44	53.13
0.90	−11.64	43.45	−7.81	62.18	−4.85	70.62
1.00	−13.98	90.00	−7.96	90.00	−4.44	90.00
1.10	−10.34	133.67	−6.24	115.51	−3.19	107.65
1.20	−6.00	151.39	−3.73	132.51	−1.48	121.43
1.30	−2.65	159.35	−1.27	143.00	0.35	131.50
1.40	0.00	163.74	0.92	149.74	2.11	138.81
1.50	2.18	166.50	2.84	154.36	3.75	144.25
1.60	4.04	168.41	4.54	157.69	5.26	148.39
1.70	5.67	169.80	6.06	160.21	6.64	151.65
1.80	7.12	170.87	7.43	162.18	7.91	154.26
1.90	8.42	171.72	8.69	163.77	9.09	156.41
2.00	9.62	172.41	9.84	165.07	10.19	158.20
3.00	18.09	175.71	18.16	171.47	18.28	167.32
4.00	23.53	176.95	23.57	173.91	23.63	170.91
5.00	27.61	177.61	27.63	175.24	27.67	172.87
6.00	30.89	178.04	30.90	176.08	30.93	174.13
7.00	33.63	178.33	33.64	176.66	33.66	175.00
8.00	35.99	178.55	36.00	177.09	36.01	175.64
9.00	38.06	178.71	38.07	177.42	38.08	176.14
10.00	39.91	178.84	39.92	177.69	39.93	176.53

- $\boldsymbol{\omega_n}$ is the break frequency for the second order polynomial.

Learning Unit 10 – Frequency Response Techniques

- **Bode Plots for $\boldsymbol{G(s) = \frac{1}{s^2+2\zeta\omega_n s+\omega_n{}^2}}$**

$$G(j\omega) = \frac{1}{(j\omega)^2+2j\zeta\omega_n\omega + \omega_n{}^2}$$

$$G(j\omega) = \frac{1}{\omega_n{}^2\left(-\left(\frac{\omega}{\omega_n}\right)^2 + 2j\zeta\,\frac{\omega}{\omega_n} + 1\right)}$$

<u>Magnitude M (dB) :</u>

$$20\ \log|G(s)| = -20\ \log\,\omega_n{}^2 - 20\ \log\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\,\frac{\omega}{\omega_n}\right)^2}$$

<u>Angle :</u> $\varphi = -tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)$

Learning Unit 10 – Frequency Response Techniques

$$G(j\omega) = \frac{1}{{\omega_n}^2\left(-\left(\frac{\omega}{\omega_n}\right)^2 + 2j\zeta\frac{\omega}{\omega_n} + 1\right)}$$

M (dB) $= -20\log{\omega_n}^2 - 20\log\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}$, $\varphi = -tan^{-1}\left(\frac{2\zeta\frac{\omega}{\omega_n}}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right)$

Low frequency : $\boldsymbol{\omega \ll \omega_n, \omega \to 0}$.

Magnitude (dB)$= -20\log{\omega_n}^2 = -40\log\ \omega_n$

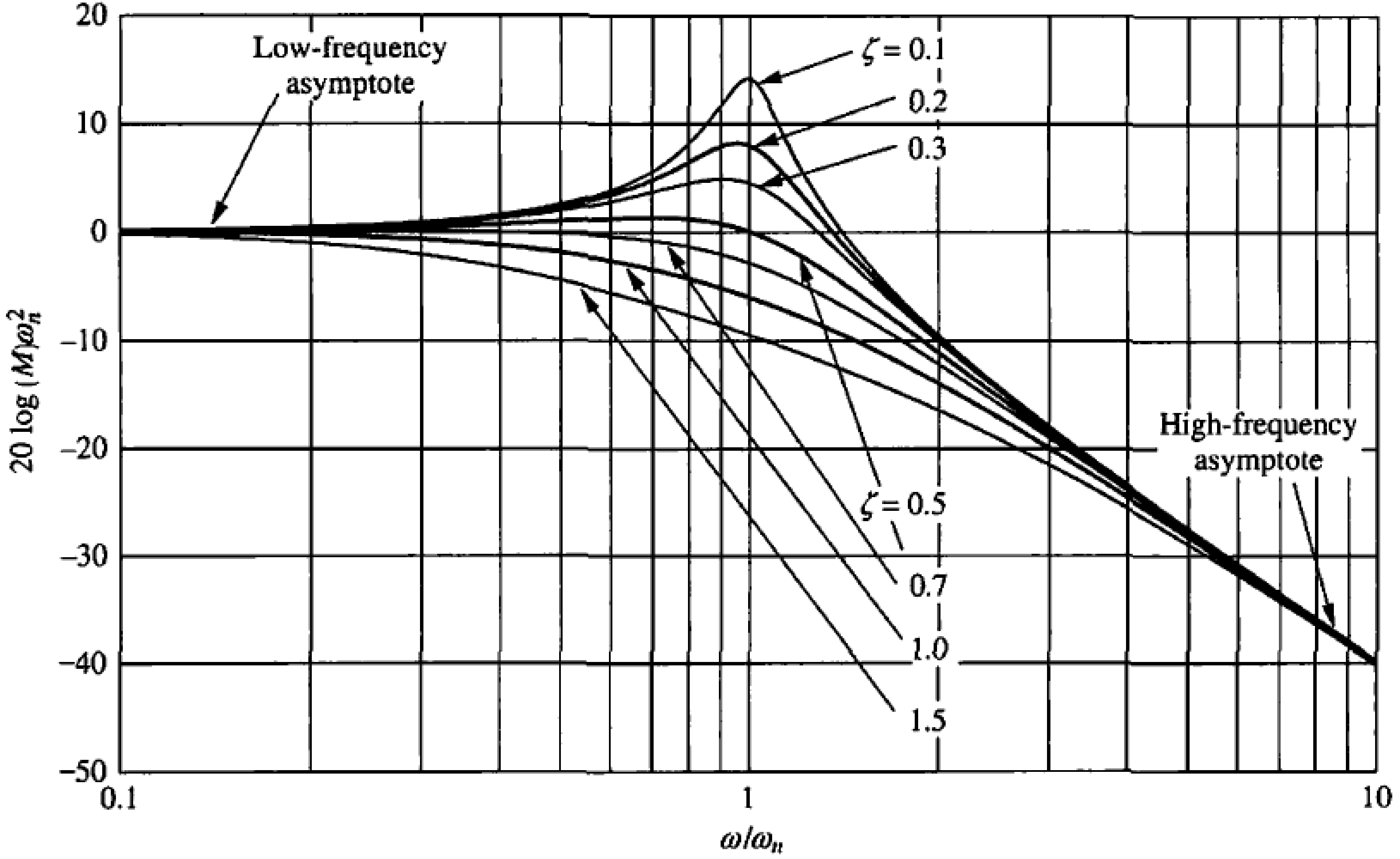
$\varphi = tan^{-1}(0) = 0°$

High Frequency : : $\boldsymbol{\omega \gg \omega_n,\ \frac{\omega}{\omega_n} \gg 1, \omega \to \infty,}$

M (dB)$= -20\log{\omega_n}^2 - 20\log\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2} \approx$

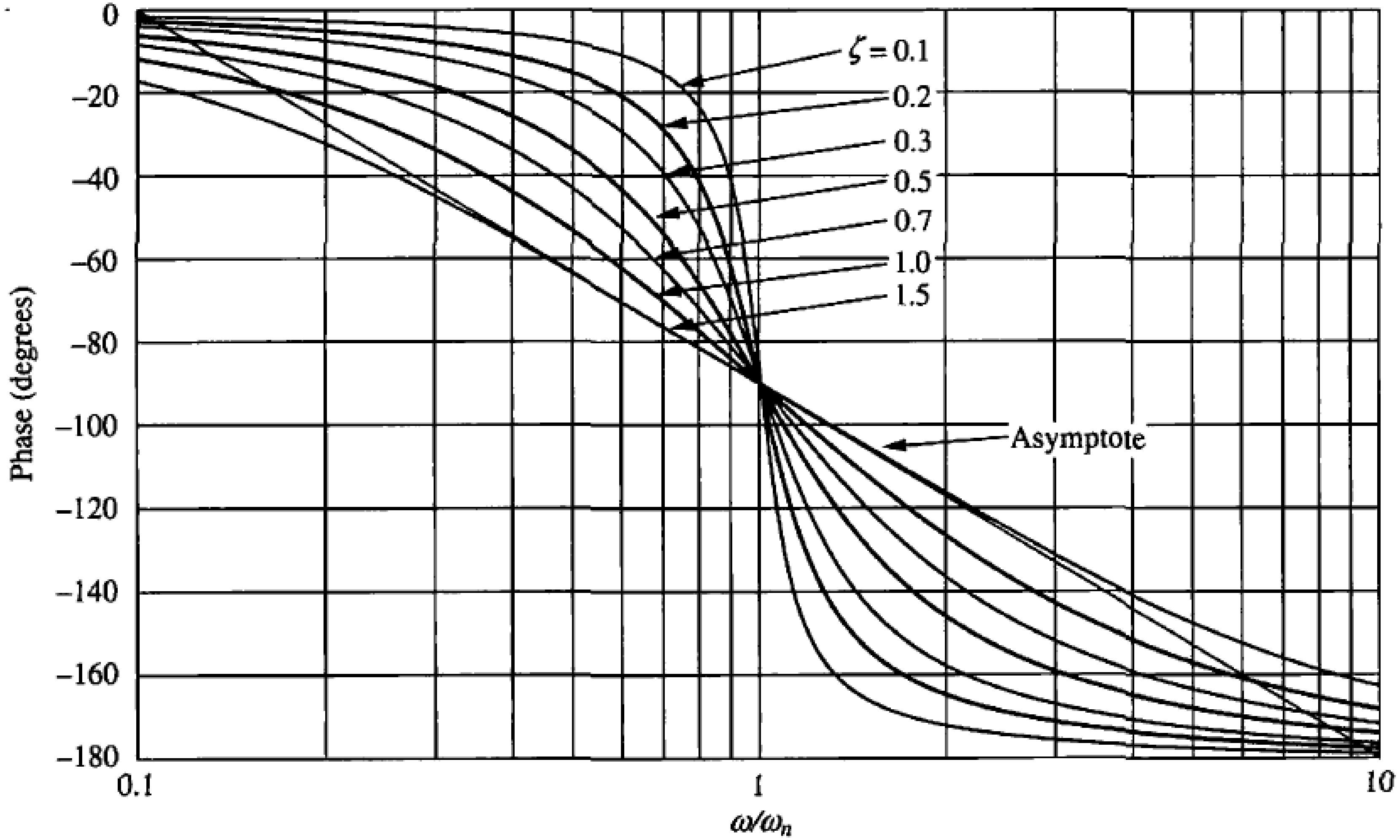
Magnit$\approx -20\log{\omega_n}^2 - 20log\left(\frac{\omega}{\omega_n}\right)^2 =$ **$-$40 log $\boldsymbol{\omega}$**

$\boldsymbol{G(j\omega) \approx -\frac{1}{\omega^2} = \frac{1}{\omega^2}\angle -180°}$



Normalized and scaled log-magnitude response for $G(s) = 1/(s^2 + 2\zeta\omega_n s + \omega_n^2)$

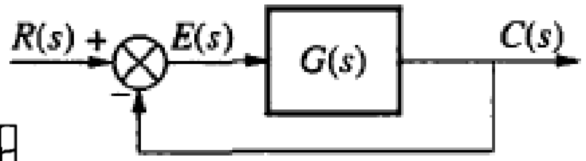
Learning Unit 10 – Frequency Response Techniques

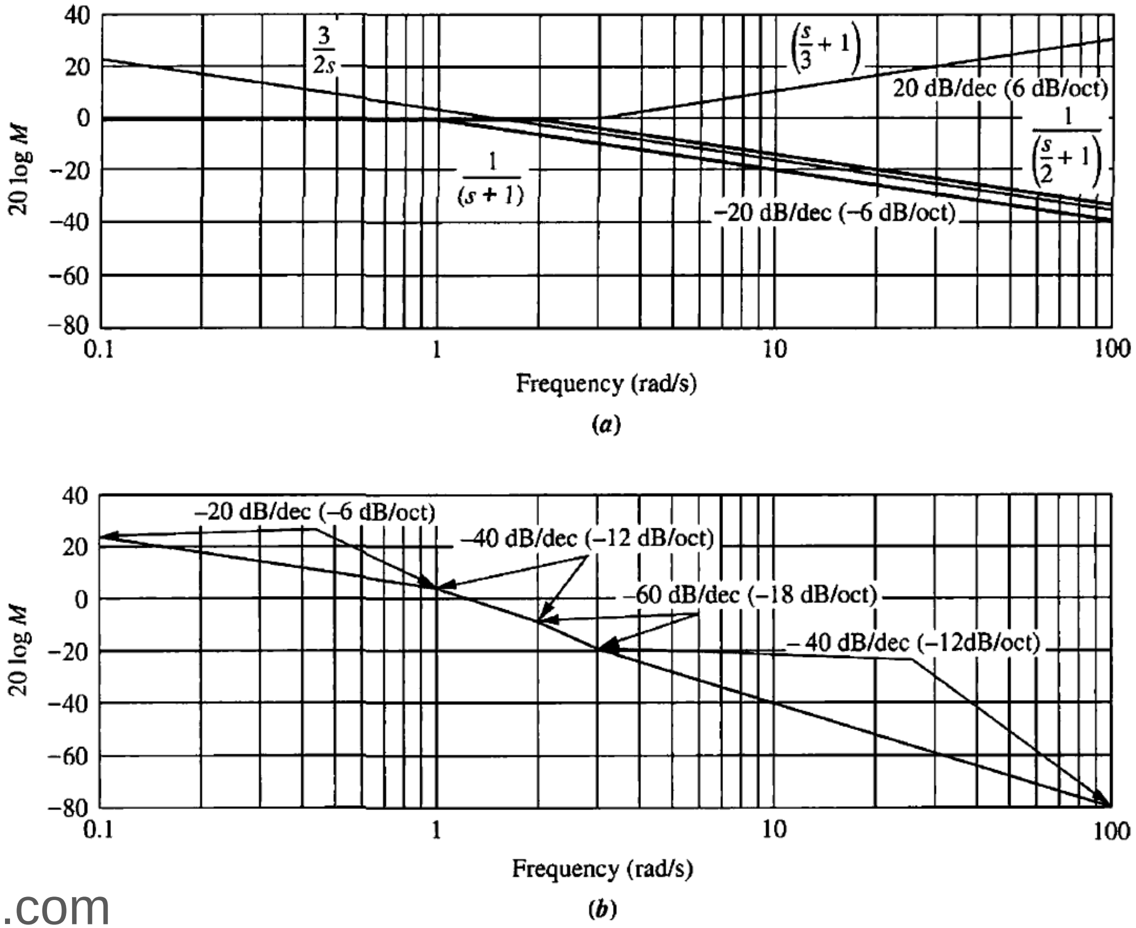


Scaled phase response for $G(s) = 1/(s^2 + 2\zeta\omega_n s + \omega_n{}^2)$

Learning Unit 10 – Frequency Response Techniques

- Example: **Bode Plots for Ratio of First-Orde**





$$G(s) = \frac{(s+3)}{s(s+1)(s+2)}$$

Learning Unit 10 – Frequency Response Techniques

	Frequency (rad/s)			
Description	**0.1 (Start: Pole at 0)**	**1 (Start: Pole at −1)**	**2 (Start: Pole at −2)**	**3 (Start: Zero at −3)**
Pole at 0	−20	−20	−20	−20
Pole at −1	0	−20	−20	−20
Pole at −2	0	0	−20	−20
Zero at −3	0	0	0	20
Total slope (dB/dec)	−20	−40	−60	−40

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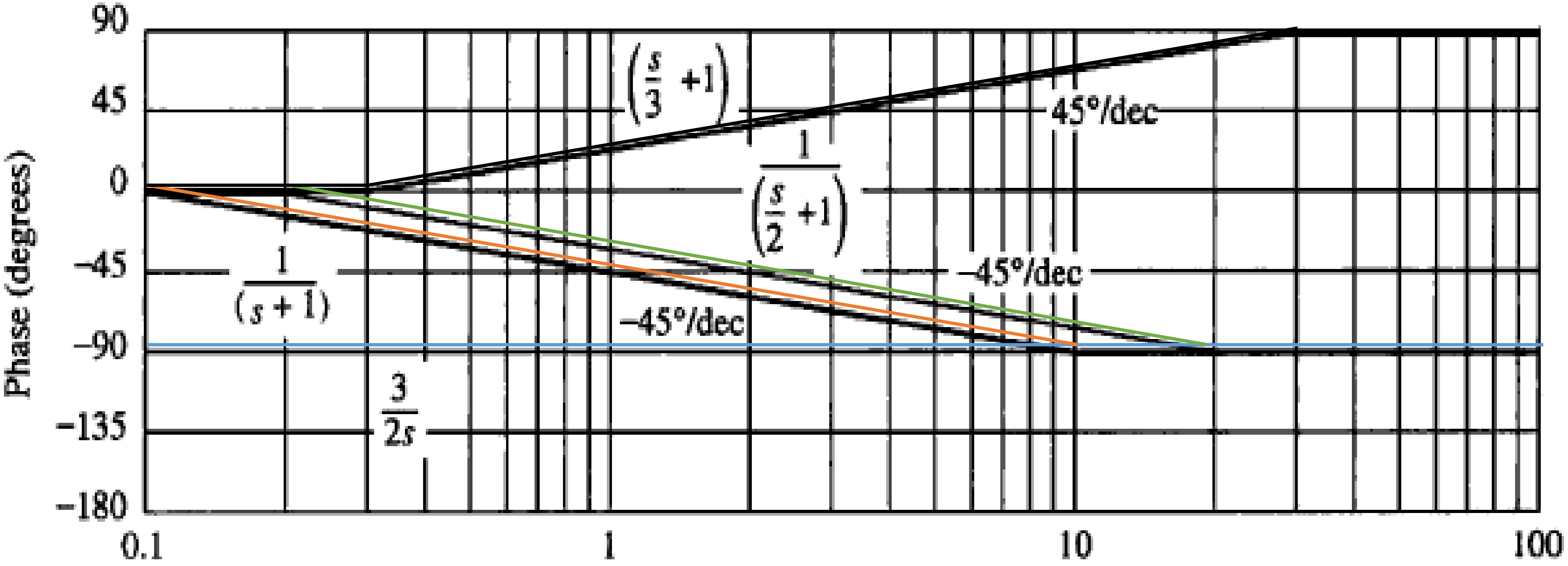
$$G(s) = \frac{(s+3)}{s(s+1)(s+2)} = \frac{3\left(\frac{s}{3}+1\right)}{2\,s\,(s+1)\left(\frac{s}{2}+1\right)}$$



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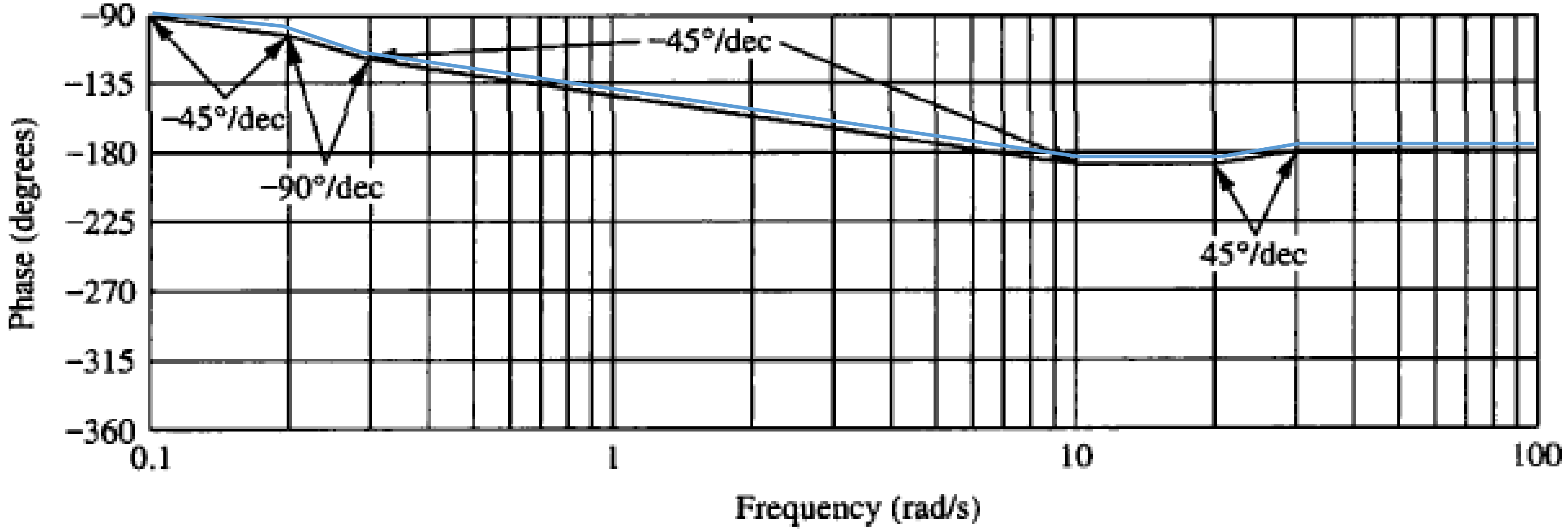
Description	Frequency (rad/s)					
	0.1 (Start: Pole at −1)	0.2 (Start: Pole at −2)	0.3 (Start: Pole at −3)	0 (End: Pole at −1)	20 (End: Pole at −2)	30 (End: Zero at −3)
Pole at −1	−45	−45	−45	0		
Pole at −2		−45	−45	−45	0	
Zero at −3			45	45	45	0
Total slope (deg/dec)	−45	−90	−45	0	45	0



Learning Unit 10 – Frequency Response Techniques

- **Example: Bode Plots for Ratio of First- and Second-Order Factors**

$$G(s) = \frac{s+3}{(s+2)(s^2+2s+25)}$$

$$G(s) = \frac{3}{(2)(25)} \frac{\left(\frac{s}{3}+1\right)}{\left(\frac{s}{2}+1\right)\left(\frac{s^2}{25}+\frac{2}{25}s+1\right)} = \frac{3}{50} \frac{\left(\frac{s}{2}+1\right)}{\left(\frac{s}{2}+1\right)\left(\frac{s^2}{25}+\frac{2}{25}s+1\right)}$$

	Frequency (rad/s)			
Description	**0.01 (Start: Plot)**	**2 (Start: Pole at −2)**	**3 (Start: Zero at −3)**	**5 (Start: $\omega_n = 5$)**
Pole at −2	0	−20	−20	−20
Zero at −3	0	0	20	20
$\omega_n = 5$	0	0	0	−40
Total slope (dB/dec)	0	−20	0	−40

Magnitude diagram slopes

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	Frequency (rad/s)					
Description	**0.2 (Start: Pole at −2)**	**0.3 (Start: Zero at −3)**	**0.5 (Start: ω_n at −5)**	**20 (End: Pole at −2)**	**30 (End: Zero at −3)**	**50 (End: $\omega_n = 5$)**
Pole at −2	−45	−45	−45	0		
Zero at −3		45	45	45	0	
$\omega_n = 5$			−90	−90	−90	0
Total slope (dB/dec)	−45	0	−90	−45	−90	0

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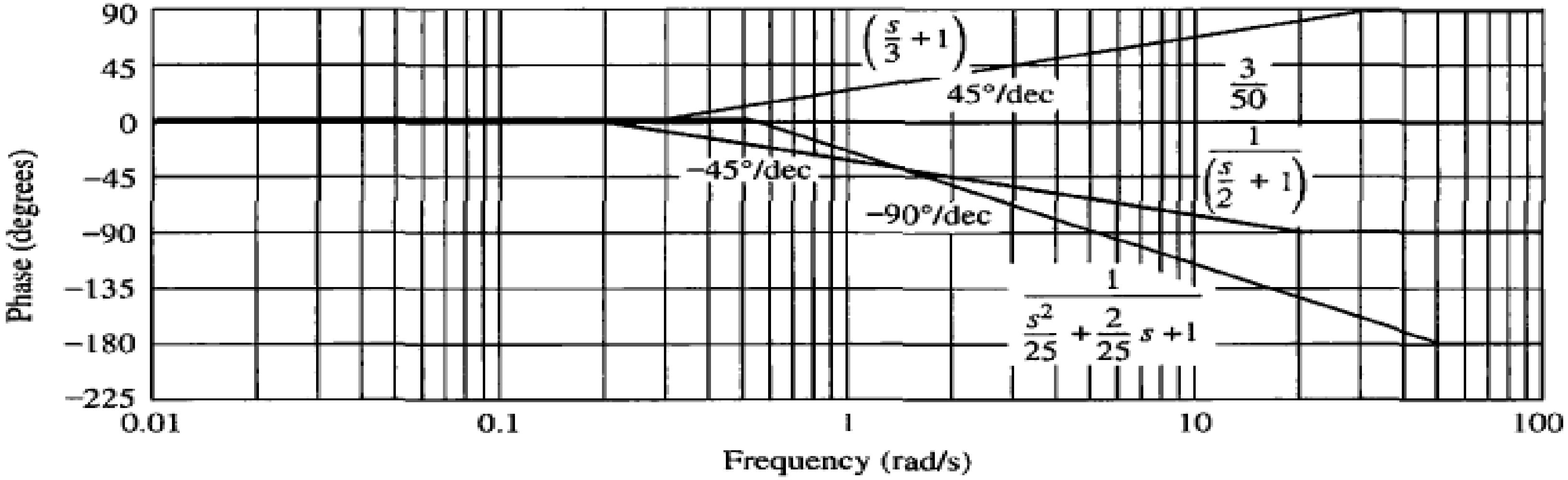
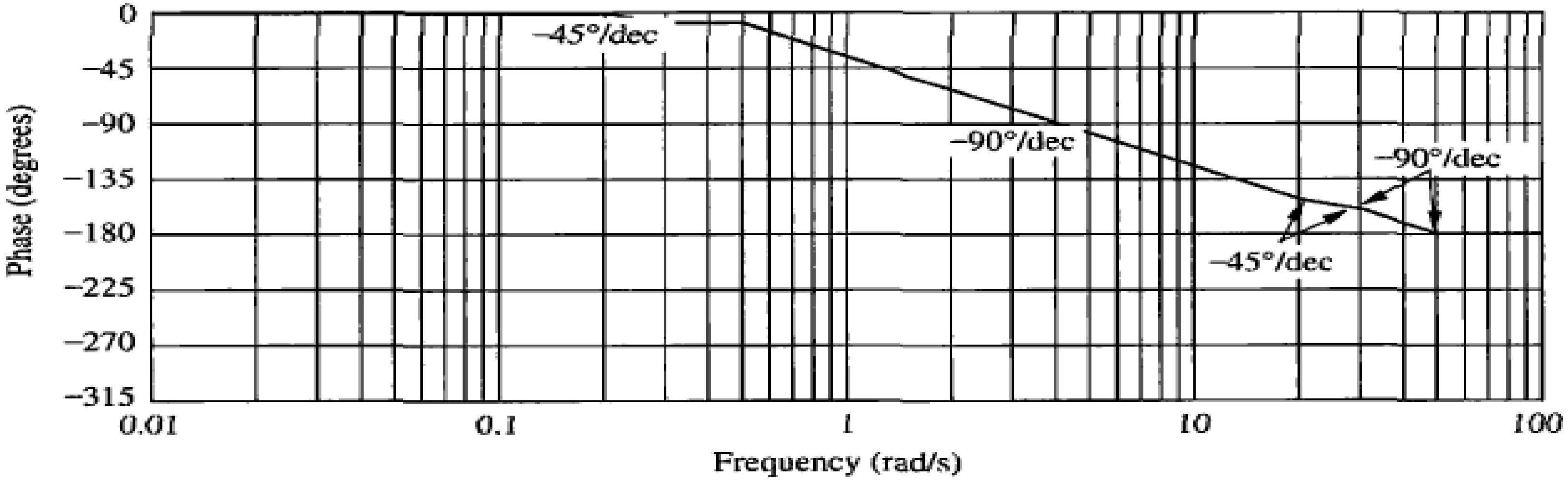
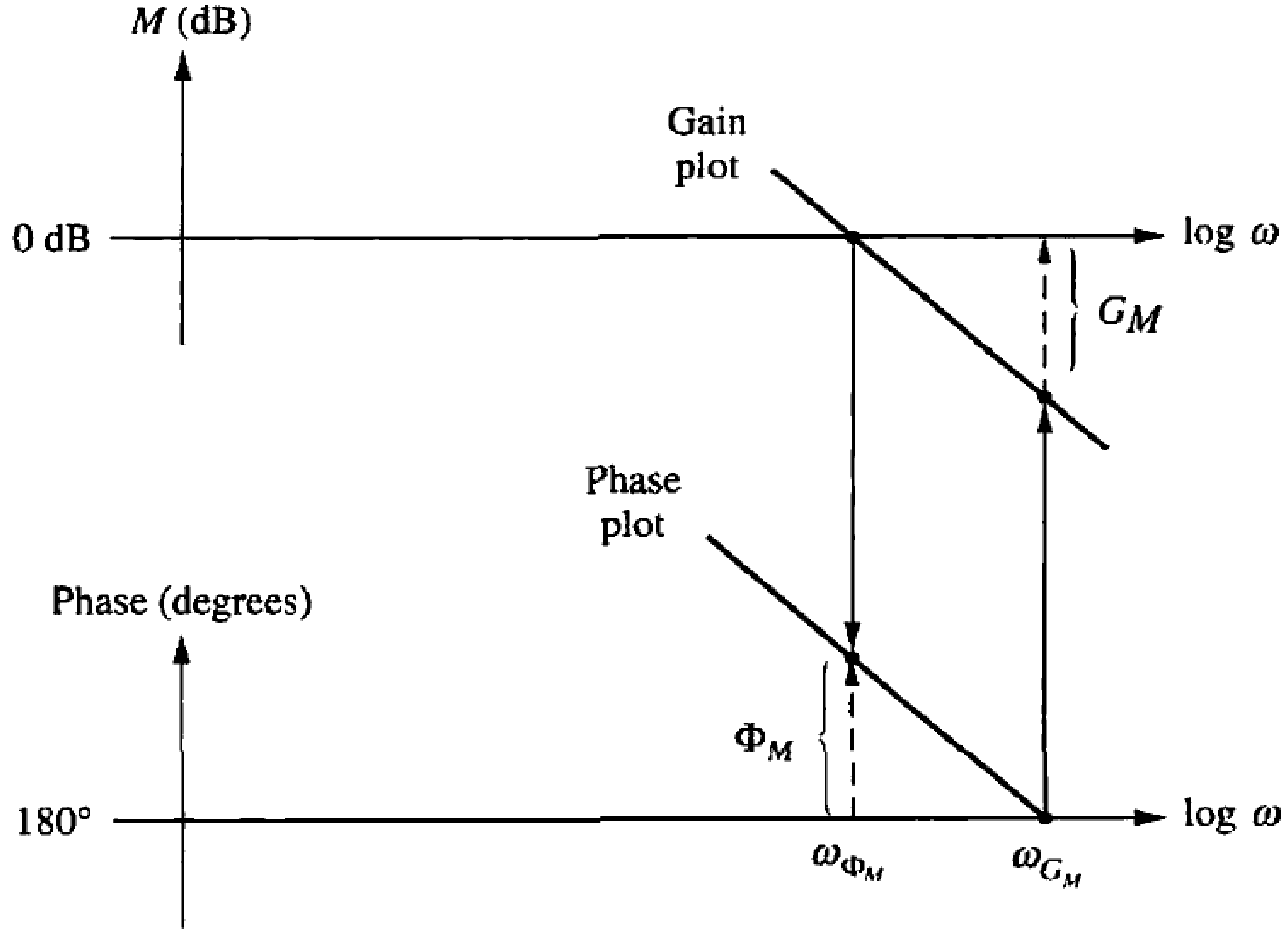


(a)

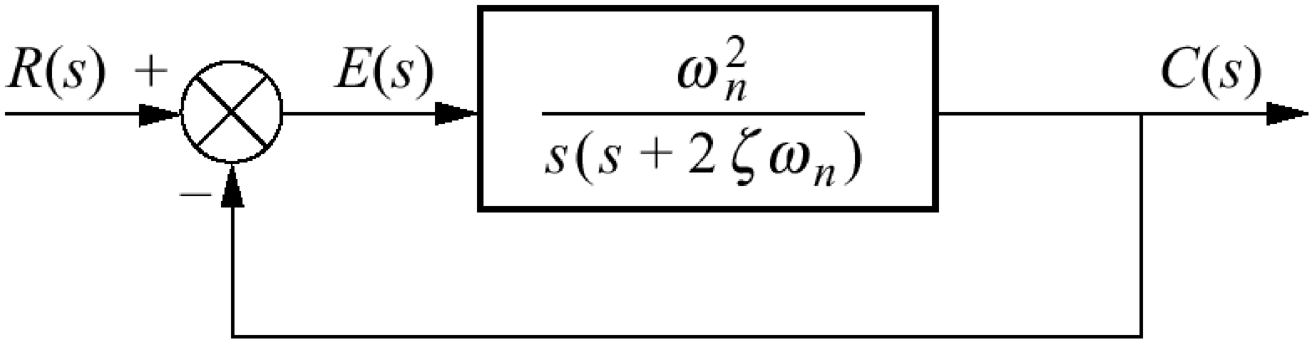


(b)

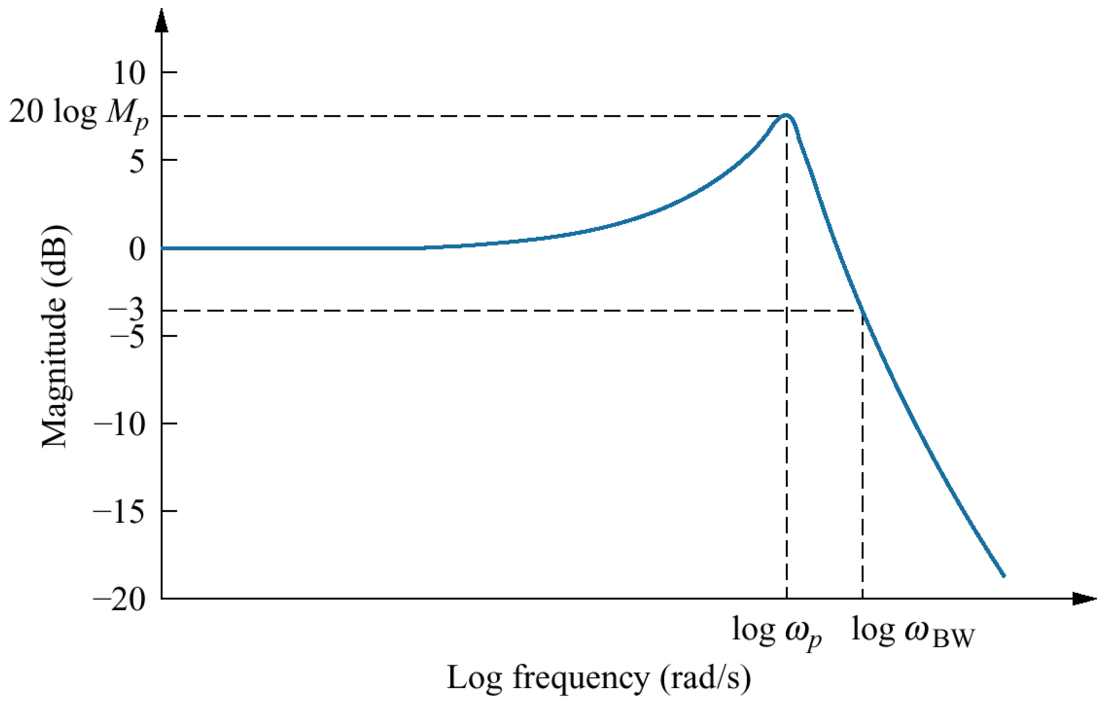


Gain and phase margins on the Bode diagrams

Second-order closed-loop system



Representative log-magnitude plot of Eq. (10.51)



$$M = |T(j\omega)| = \frac{\omega_n^2}{\sqrt{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}}$$

$$M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

$$\omega_p = \omega_n\sqrt{1-2\zeta^2}$$